“RANKING INTERSECTING LORENZ CURVES”

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Ranking Intersecting Lorenz Curves

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This paper is concerned with the problem of ranking Lorenz curves in situations where the Lorenz curves intersect and no unambiguous ranking can be attained without introducing weaker ranking criteria than first-degree Lorenz dominance. To deal with such situations two alternative sequences of nested dominance criteria between Lorenz curves are introduced. At the limit the systems of dominance criteria appear to depend solely on the income share of either the worst-off or the best-off income recipient. This result suggests two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered; one that places more emphasis on changes that occur in the lower part of the income distribution and the other that places more emphasis on changes that occur in the upper part of the income distribution. Both strategies turn out to depart from the Gini coefficient; one requires higher degree of downside and the other higher degree of upside inequality aversion than what is exhibited by the Gini coefficient. Furthermore, it is demonstrated that the sequences of dominance criteria characterize two separate systems of nested subfamilies of inequality measures and thus provide a method for identifying the least restrictive social preferences required to reach an unambiguous ranking of a given set of Lorenz curves. Moreover, it is demonstrated that the introduction of successively more general transfer principles than the Pigou-Dalton principle of transfers and more restrictive transformations than the mean-preserving spread (contractions) forms a helpful basis for judging the normative significance of higher degrees of Lorenz dominance. Journal of Economic Literature Classification Numbers: D31, D63.

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1. INTRODUCTION

In empirical analyses of income distributions it is common practice to make separate comparisons of mean incomes and Lorenz curves. The Lorenz curve, which was introduced by Lorenz [30] as a representation of inequality, is concerned with income shares without taking account of differences in mean incomes. By displaying the deviation of each individual income share from the income share that corresponds to perfect equality, the Lorenz curve captures the essential descriptive features of the concept of inequality. Thus, adopting the Lorenz curve as a basis for judging between income distributions means that we focus solely on distributional aspects. The widespread use of the Lorenz curve in applied work shows that focusing on distributional aspects is of interest in its own right\(^2\), irrespective of how we judge between level of mean income and degree of inequality in cases where they conflict. For welfare judgments about the trade-off between mean income and inequality we refer to Shorrocks [40], Ebert [16] and Lambert [27, 28].

Ranking Lorenz curves in accordance with first-degree Lorenz dominance means that the higher of non-intersecting Lorenz curves is preferred. The normative significance of this criterion follows from the fact that the higher of two non-intersecting Lorenz curves can be obtained from the lower Lorenz curve by means of rank-preserving income transfers from richer to poorer individuals, which means that the criterion of first-degree Lorenz dominance is consistent with the Pigou-Dalton principle of transfers. Thus, when one Lorenz curve lies above another the higher Lorenz curve displays less inequality than the lower Lorenz curve. However, since Lorenz curves may intersect, which is often the case in applied economics, other ranking criteria than first-degree Lorenz dominance are needed to reach an unambiguous conclusion.

The standard practice for ranking intersecting Lorenz curves is to apply summary measures of inequality. However, as it may be difficult to find a single measure that gains a wide degree of support, it is of interest to search for alternative ranking criteria that are stronger than single measures of inequality and weaker than first-degree Lorenz dominance. To this end two alternative dominance criteria emerge as natural candidates; one that aggregates the Lorenz curve from below (second-degree upward Lorenz dominance) and the other that aggregates the Lorenz curve from above (second-degree downward Lorenz dominance). Since first-degree Lorenz dominance implies second-degree upward as well as downward Lorenz dominance we have that both methods preserve first-degree Lorenz dominance and thus are consistent with the Pigou-Dalton principle of transfers. However, the transfer sensitivity of these criteria differ in the sense that second-degree upward Lorenz dominance place more emphasis on transfers occurring in the lower rather than in the upper part of the income distribution, whereas second-degree downward Lorenz dominance is most sensitive to transfers that occur in the upper part of the

\(^2\) See e.g. Atkinson et al. [4] who make cross-country comparisons of Lorenz curves allowing for differences between countries in level of income and Lambert [29] for a discussion of applying Lorenz dominance criteria as basis for evaluating distributional effects of tax reforms.
income distribution. This means that the criterion of second-degree upward Lorenz dominance requires a transfer of money from a richer to a poorer person to be more equalizing the lower it occurs in the income distribution, provided that the proportion of individuals between the donors and receivers is fixed. By contrast, the criterion of second-degree downward Lorenz dominance requires this type of transfer to be more equalizing the higher it occurs in the income distribution.

The relationship between first- and second-degree upward Lorenz dominance and measurement of inequality has been widely discussed in the economic literature. Restricting attention to distributions of equal means, Kolm [24] and Atkinson [3] observed that first-degree Lorenz dominance and second-degree stochastic dominance are identical requirements, and thus recognized that the family of inequality measures derived from utilitarian social welfare functions with concave utility functions yields a characterization of the criterion of non-intersecting Lorenz curves. This result suggests the hypothesis that second-degree upward Lorenz dominance imposes the restriction of positive third derivative on the utility function of the utilitarian inequality measures, where second-degree upward Lorenz dominance is defined analogous to second-degree stochastic dominance. Unfortunately, this hypothesis has to be rejected since second-degree upward Lorenz dominance and third-degree stochastic dominance do not coincide. However, useful analyses of the implications of third-degree stochastic dominance on measurement of inequality and social welfare have been provided by Shorrocks and Foster [41], Dardanoni and Lambert [10] and Davis and Hoy [12, 13], whilst Muliere and Scarsini [33] and Zoli [49, 50] have examined the implications of applying second-degree Lorenz dominance as a criterion for ranking Lorenz curves.

While the majority of the results in these papers concerns the case of singly intersecting Lorenz curves the latter four papers provide results for the case of multiple crossings as well. However, the ranking criterion introduced by Davis and Hoy requires computation and comparison of the coefficient of variation for each of the actual intersections between the Lorenz curves being compared. The complexity of this approach may be considered as a drawback and makes it less attractive as a practical method for ranking Lorenz curves. By contrast, the results of Muliere and Scarsini [33] and Zoli [49, 50] suggest that there may be a closer relationship between Lorenz dominance (of various degrees) and rank-dependent measures of inequality than between Lorenz dominance and utilitarian measures of inequality. This is due to the fact that rank-dependent measures of inequality, as apposed to utilitarian measures of inequality, are explicitly defined in terms of the Lorenz curve. The most widely known rank-dependent measure of inequality is the Gini coefficient, which summarizes the deviations of the observed individual income shares and the corresponding income shares under the condition of perfect equality. As proposed by Mehran [31] we may use weighed sums of the income share deviations as alternative rank-dependent measures of inequality to the Gini coefficient. The family of rank-dependent measures of inequality, which includes the Gini

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coefficient, can alternatively be expressed as a weighted area between the Lorenz curve \( L(u) \) and its equality reference \( u \). The chosen specification of the rank-dependent weight-function, which may be considered as the preference function of a social decision-maker, clarifies whether concern about inequality is particularly related to the lower, the central or the upper part of the income distribution. Thus, the functional form of the weight-function exhibits the inequality aversion profile of the corresponding measure of inequality. Roughly speaking, we may say that a rank-dependent measure of inequality exhibits downside inequality aversion when the weight-function gives more emphasis to the deviation between the Lorenz curve and its equality reference, \( u - L(u) \), for small \( u \) than for large \( u \). By contrast, when the weight-function gives more weight to the deviation \( u - L(u) \) for large \( u \) than for small \( u \), we may say that the corresponding inequality measure exhibits upside inequality aversion.

The purpose of this paper is to explore what restrictions various Lorenz dominance criteria place on the weight-functions of the family of rank-dependent measures of inequality, and to provide a normative justification of these criteria by introducing appropriate principles of transfer sensitivity and transformations that can be considered as generalizations of the mean-preserving-spread. As will be demonstrated in Sections 2 and 3, second-degree Lorenz dominance forms a natural basis for the construction of two separate hierarchical sequences of partial orderings (dominance criteria), where one sequence places emphasize on changes that occur in the lower part of the Lorenz curve whereas the other places emphasize on changes that occur in the upper part of the Lorenz curve. The hierarchical and nested structure of the dominance criteria appears to be useful in empirical applications since we are allowed to identify the lowest degree of dominance required to reach unambiguous rankings of Lorenz curves. Moreover, Section 3 demonstrates that the two hierarchical sequences of Lorenz dominance criteria divide the family of rank-dependent measures of inequality into two corresponding hierarchical systems of nested subfamilies that offer two different inequality aversion profiles; one exhibits successively higher degrees of downside inequality aversion whereas the other exhibits successively higher degrees of upside inequality aversion. Since the criteria of Lorenz dominance provide convenient computational methods, these results can be used to identify the largest subfamily of the family of rank-dependent measures of inequality and thus the least restrictive social preferences required to reach unambiguous ranking of any given set of Lorenz curves. Section 4 uses these characterization results to arrange the members of two different generalized Gini families of inequality measures into subfamilies according to their relationship to Lorenz dominance of various degrees. Section 5 summarizes the conclusions of the paper and briefly discusses the use of the obtained results as a basis for deriving dominance criteria for generalized Lorenz curves.

2. LORENZ DOMINANCE OF FIRST AND SECOND DEGREE

The Lorenz curve \( L \) for a cumulative income distribution \( F \) with mean \( \mu \) is defined by
\[ L(u) = \frac{1}{\mu} \int_0^u F^{-1}(t) \, dt, \quad 0 \leq u \leq 1, \quad (1) \]

where \( F^{-1}(t) = \inf \{ x : F(x) \geq t \} \) is the left inverse of \( F \). Thus, the Lorenz curve \( L(u) \) shows the share of total income received by the poorest 100 \( u \) per cent of the population. Note that \( F \) can either be a discrete or a continuous distribution function. Although the former is what we actually observe, the latter often allows simpler derivation of theoretical results and is a valid large sample approximation. Thus, in most cases below \( F \) will be assumed to be a continuous distribution function, but the assumption of a discrete distribution function will be used where appropriate.

Under the restriction of equal mean incomes the problem of ranking Lorenz curves formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Kolm [24] and Atkinson [3] to characterize the criterion of non-intersecting Lorenz curves in the case of distributions with equal mean incomes. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance\(^4\), which means that the criterion of non-intersecting Lorenz curves obeys the Pigou-Dalton principle of transfers. The Pigou-Dalton principle of transfers states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged, and is defined formally by\(^5\)

**DEFINITION 2.1. (The Pigou-Dalton principle of transfers.)** Consider a discrete income distribution \( F \). A transfer \( \delta \) from a person with income \( F^{-1}(t) \) to a person with income \( F^{-1}(s) \), where the transfer is assumed to be rank-preserving, is said to reduce inequality in \( F \) when \( s < t \) and raise inequality in \( F \) when \( s > t \). The former transfer will be denoted a Pigou-Dalton progressive transfer and the latter transfer a Pigou-Dalton regressive transfer.

To perform inequality comparisons with Lorenz curves we can deal with distributions with equal means, or alternatively simply abandon the assumption of equal means and consider distributions of relative incomes.\(^6\) The latter approach normally forms the basis of empirical studies.

The standard criterion of non-intersecting Lorenz curves, called first-degree Lorenz dominance, is based on the following definition\(^7\).

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\(^4\) For a proof see Hardy, Littlewood and Polya [23].

\(^5\) Note that this definition of the Pigou-Dalton principle of transfers was proposed by Fields and Fei [18].

\(^6\) The importance of focusing on relative incomes was acknowledged already by Plato who proposed that the ratio of the top income to the bottom should be less than four to one (see Cowell, [9]). See also Sen's [38] discussion of relative deprivation and Smith's [42] discussion of necessities.

\(^7\) Note that most analyses of Lorenz dominance apply a definition that excludes the requirement of strict inequality for some \( u \).
DEFINITION 2.2. A Lorenz curve $L_1$ is said to \textit{first-degree dominate} a Lorenz curve $L_2$ if

$$L_1(u) \geq L_2(u) \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in (0,1)$.

In order to examine the relationship between various Lorenz dominance criteria and the measurement of inequality we will rely on the \textit{family of rank-dependent measures of inequality} defined by

$$J_P(L) = 1 - \frac{1}{\mu} \int_0^1 P'(u) dL(u) = 1 - \frac{1}{\mu} \int_0^1 P'(u) F^{-1}(u) du \quad (2)$$

where $L$ is the Lorenz curve of the income distribution $F$ with mean $\mu$ and the weight-function $P'$ is the derivative of a continuous, differentiable and concave function $P$ defined on the unit interval where $P(0) = 0$ and $P(1) = 1$. To ensure that $J_P$ has the unit interval as its range the condition $P'(1) = 0$ is imposed on $P$. As demonstrated by Yaari [46, 47] and Aaberge [2] the $J_P$-family represents a preference relation defined either on the class of distribution functions ($F$) or on the class of Lorenz curves ($L$), where $P$ can be interpreted as a preference function of a social decision-maker. The preference function $P$ assigns weights to the incomes of the individuals in accordance with their rank in the income distribution. Therefore, the functional form of $P$ reveals the attitude towards inequality of a social decision-maker who employs $J_P$ to judge between Lorenz curves. The most well-known member of the $J_P$-family is the Gini coefficient, which is defined by

$$G(L) = 1 - 2 \int_0^1 L(u) du = 1 - 2 \int_0^1 \left(1 - u\right) F^{-1}(u) du \quad (3)$$

\[8\] Mehran [31] introduced the $J_P$-family by relying on descriptive arguments. A slightly different version of $J_P$ was introduced by Piesch [34], whereas Giaccardi [21] considered a discrete version of $J_P$. For alternative normative motivations of the $J_P$-family and various subfamilies of the $J_P$-family we refer to Donaldson and Weymark [14, 15], Weymark [44], Yaari [46, 47], Ben Porath and Gilboa [5], Aaberge [2] and Gajdos [20].
As demonstrated by Yaari [47], the \( J_P \)-family of inequality measures can be used as a basis for characterizing first-degree Lorenz dominance. For the sake of completeness the characterization result of first-degree Lorenz dominance given by Yaari [47] is reproduced in Theorem 2.1 below, where \( L \) denotes the family of Lorenz curves and \( P_1 \) is a class of preference functions defined by

\[
P_1 = \{ P : P' \text{ and } P'' \text{ are continuous on } [0,1], P'(t) > 0 \text{ and } P''(t) < 0 \text{ for } t \in (0,1), \text{ and } P'(1) = 0 \}.
\]

THEOREM 2.1. (Fields and Fei [18] and Yaari [47]). Let \( L_1 \) and \( L_2 \) be members of \( L \). Then the following statements are equivalent,

(i) \( L_1 \) first-degree dominates \( L_2 \)
(ii) \( L_1 \) can be obtained from \( L_2 \) by a sequence of Pigou-Dalton progressive transfers
(iii) \( L_2 \) can be obtained from \( L_1 \) by a sequence of Pigou-Dalton regressive transfers
(iv) \( J_P(L_1) < J_P(L_2) \) for all \( P \in P_1 \)

We refer to Fields and Fei [18] for a proof of the equivalence between (i) and (ii) (and (iii))\(^9\) and to Yaari [47] for a proof of the equivalence between (i) and (iv).

Atkinson [3] defined inequality aversion as equivalent to risk aversion in the theory of choice under uncertainty. This was motivated by the fact that the Pigou-Dalton transfer principle is identical to the principle of mean-preserving spread introduced by Rothschild and Stiglitz [35], which is equivalent to the condition of dominating non-intersecting Lorenz curves. Thus, we adopt the following definition.

DEFINITION 2.3. A social decision-maker that supports the Pigou-Dalton principle of transfers is said to exhibit inequality aversion.

A social decision-maker who prefers the dominating one of non-intersecting Lorenz curves favors transfers of incomes which reduce the differences between the income shares of the donor and the recipient, and is therefore said to be inequality averse.

The characterization of the condition of first-degree Lorenz dominance provided by Theorem 2.1 shows that non-intersecting Lorenz curves can be ordered without specifying further the functional form of the preference function \( P \) other than \( P \) being strictly concave. This means that \( J_P \) satisfies the Pigou-Dalton principle of transfers for concave \( P \)-functions. To deal with situations where Lorenz curves intersect a weaker principle than first-degree Lorenz dominance is called for. To this end we may employ second-degree upward Lorenz dominance defined by

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\(^9\) See Rothschild and Stiglitz [36] for a proof of the equivalence between (i), (ii) and (iii) in the case where the rank-preserving condition is abandoned in the definition of the Pigou-Dalton principle of transfers.
DEFINITION 2.4A. A Lorenz curve \( L_1 \) is said to **second-degree upward dominate** a Lorenz curve \( L_2 \) if

\[
\int_0^u L_1(t) \, dt \geq \int_0^u L_2(t) \, dt \quad \text{for all} \quad u \in [0,1]
\]

and the inequality holds strictly for some \( u \in \{0,1\} \).

The term upward dominance refers to the fact that the Lorenz curves are aggregated from below\(^{10}\). The aggregated Lorenz curve can be considered as a sum of weighted income shares, where the weights decrease linearly with increasing rank of the income receiver in the income distribution. Thus, a social decision-maker who prefers the second-degree upward dominating of two intersecting Lorenz curves pays more attention to inequality in the lower than in the upper part of the income distribution. An alternative ranking criterion to second-degree upward Lorenz dominance can be obtained by aggregating the Lorenz curve from above.

DEFINITION 2.4B. A Lorenz curve \( L_1 \) is said to **second-degree downward dominate** a Lorenz curve \( L_2 \) if

\[
\int_u^1 (1-L_2(t)) \, dt \geq \int_u^1 (1-L_1(t)) \, dt \quad \text{for all} \quad u \in [0,1]
\]

and the inequality holds strictly for some \( u \in \{0,1\} \).

Note that second-degree downward as well as upward Lorenz dominance preserves first-degree Lorenz dominance since first-degree Lorenz dominance implies second-degree upward as well as second-degree downward Lorenz dominance. Consequently, both dominance criteria are consistent with the Pigou-Dalton principle of transfers. The choice between second-degree upward and downward Lorenz dominance clarifies whether or not equalizing transfers between poorer individuals should be considered more important than those between richer individuals. A social decision-maker who favors second-degree upward Lorenz dominance would most likely prefer third-degree upward Lorenz dominance to third-degree downward Lorenz dominance, because third-degree upward Lorenz dominance places the emphasis on equalizing transfers between poorer individuals, whereas third-degree downward Lorenz dominance places the emphasis on equalizing transfers between richer individuals.

\(^{10}\) Note that second-degree upward Lorenz dominance is equivalent to a normalized version of third-degree inverse stochastic dominance introduced by Muliere and Scarsini (1989).
As recognized by Muliere and Scarsini [33] there is no simple relationship between third-degree stochastic dominance and second-degree upward Lorenz dominance, but that second-degree upward Lorenz dominance is equivalent to third-degree upward inverse stochastic dominance. Thus, a general characterization of second-degree Lorenz dominance or third-degree inverse stochastic dominance in terms of ordering conditions for the utilitarian measures introduced by Kolm [24] and Atkinson [3] cannot be obtained. As explained in Section 1 the family of rank-dependent measures of inequality (defined by (2)) appears to form a more convenient basis for judging the normative significance of second-degree and higher degrees of Lorenz dominance than the utilitarian families introduced by Kolm [24] and Atkinson [3].

To judge the normative significance of criteria for ranking intersecting Lorenz curves, more powerful principles than the Pigou-Dalton principle of transfers are needed. To this end Kolm [25, 26] introduced the principle of diminishing transfers\(^{11}\), which for a fixed difference in income considers a transfer from a richer to a poorer person to be more equalizing the further down in the income distribution it takes place.\(^{12}\) As indicated by Shorrocks and Foster [41] and Muliere and Scarsini [33] the principle of diminishing transfers is, however, not consistent with second-degree upward Lorenz dominance. Mehran [31] introduced an alternative version of the principle of diminishing transfers by accounting for the difference in the proportion of individuals between donors and receivers of the income transfers rather than for the difference in income. The principle introduced by Mehran [31] proves to characterize second-degree upward Lorenz dominance. To provide a formal definition of this principle, called the principle of positional transfer sensitivity by Zoli [49], let I be an inequality measure and let \(\Delta I(\delta,h)\) denote the change in I resulting from a transfer \(\delta\) from a person with income \(F^{-1}(t+h)\) to a person with income \(F^{-1}(t)\) that leaves their ranks in the income distribution \(F\) unchanged, where \(F\) is assumed to be a discrete distribution for a finite population. Thus, \(\Delta I(\delta,h)\) is a negative number.\(^{13}\) Furthermore, let \(\Delta I_{s,t}(\delta,h)\) be defined by

\[
\Delta I_{s,t}(\delta,h) = \Delta I(\delta,h) - \Delta I_s(\delta,h).
\]

As will become evident later it will be convenient to denote Mehran’s principle of transfers the principle of first-degree downside positional transfer sensitivity.

**DEFINITION 2.5A.** Consider a discrete income distribution \(F\) and rank-preserving transfers \(\delta\) from individuals with ranks \(s+h\) and \(t+h\) to individuals with ranks \(s\) and \(t\) in \(F\). Then the
inequality measure $I$ is said to satisfy the **principle of first-degree downside positional transfer sensitivity** (first-degree DPTS) if

$$\Delta I_{s,t}(\delta, h) > 0 \text{ when } s \neq t.$$ 

Mehran [31] demonstrated that $J_p$ defined by (2) satisfies the principle of first-degree positional transfer sensitivity if and only if $P''(t) > 0^{14}$. Moreover, as stated in Theorem 2.2A below dominance for all $J_p$ that satisfy the Pigou-Dalton principle of transfers and the principle of first-degree downside positional transfer sensitivity proves to be equivalent to the condition of second-degree upward Lorenz dominance.

The condition of second-degree downward Lorenz dominance proves to be equivalent to the condition that $P''(t) < 0$, when $J_p$ for $P \in \mathcal{P}$ is used as a ranking criterion for Lorenz curves. A social decision-maker who employs $J_p$ with $P''(t) < 0$ considers a given transfer of money from a richer to a poorer person to be more equalizing the higher it occurs in the income distribution, provided that the proportions of the population located between the receivers and the donors are equal. A formal definition of this principle, that we will call the **principle of first-degree upside positional transfer sensitivity** (first-degree UPTS), is given by

**DEFINITION 2.5B.** Consider a discrete distribution $F$ and rank-preserving transfers $\delta$ from individuals with ranks $s + h$ and $t + h$ to individuals with ranks $s$ and $t$ in $F$. Then the inequality measure $I$ is said to satisfy the principle of first-degree upside positional transfer sensitivity (first-degree UPTS) if

$$\Delta I_{s,t}(\delta, h) < 0 \text{ when } s \neq t.$$ 

Let $P_{12}^*$ be a family of preference functions related to $J_p$ and defined by

$$P_{12}^* = \{P : P \in \mathcal{P}, P'' \text{ is continuous on } [0,1] \text{ and } P''(t) > 0 \text{ for } t \in (0,1)\}.$$ 

The following result provides a characterization of second-degree upward Lorenz dominance.\(^{15}\)

**THEOREM 2.2A.** Let $L_1$ and $L_2$ be members of $\mathcal{L}$. Then the following statements are equivalent.

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\(^{14}\) Aaberge [1] demonstrated that $J_p$ defined by (2) satisfies Kolm's principle of diminishing transfers under conditions that depend on the shape of the preference function $P$ as well as on the shape of the income distribution $F$.

\(^{15}\) Note that a slightly different version of the equivalence between (i) and (ii) in Theorem 2.2A is proved by Zoli [49]. Actually, when we restrict to cases of equal means Proposition 2 of Zoli [49] and Theorem 2.2 yield identical results.
To ensure equivalence between second-degree upward Lorenz dominance and \( J_P \)-measures as decision criteria, Theorem 2.2A shows that it is necessary to restrict the preference functions \( P \) to be concave with positive third derivatives. If, by contrast, \( P \) has negative third derivative, then Theorem 2.2B yields the downward dominance analogy to Theorem 2.2A.

Let \( \mathbf{P}_{22}^* \) be a family of preference functions related to \( J_P \) and defined by

\[
\mathbf{P}_{22}^* = \{ P : P \in \mathbf{P}_1, P'''' \text{ is continuous on } [0,1] \text{ and } P''''(t) < 0 \text{ for } t \in (0,1) \}.
\]

THEOREM 2.2B. Let \( L_1 \) and \( L_2 \) be members of \( \mathbf{L} \). Then the following statements are equivalent,

(i) \( L_1 \) second-degree downward dominates \( L_2 \)

(ii) \( J_P(L_1) < J_P(L_2) \) for all \( P \in \mathbf{P}_{22}^* \)

(iii) \( J_P(L_1) < J_P(L_2) \) for all \( P \in \mathbf{P}_2 \) being such that \( J_P \) obeys the principle of first-degree UPTS.

(Proof in Appendix).

REMARK. It follows from the proofs of Theorem 2.2A and 2.2B that the condition \( P''(t) < 0 \) for all \( t \in (0,1) \) can be relaxed and replaced by the conditions \( P''(0) \leq 0 \) and \( P''(0) \leq 0 \), respectively. Moreover, note that the equivalence between (ii) and (iii) in Theorem 2.2A means that any \( J_P \) with \( P'''(t) > 0 \) obeys the first-degree DPTS.\(^{16}\) A similar remark can be made for Theorem 2.2B. Thus, any \( J_P \) with \( P'''(t) < 0 \) obeys the first-degree UPTS. However, the relevance of using measures of inequality that do not obey the Pigou-Dalton of transfers \( (P''(t) > 0) \) may be questioned.

An inequality averse social decision-maker that supports the criterion of second-degree upward Lorenz dominance will act in line with the principle of first-degree DPTS and assign more weight to changes that take place in the lower part of the Lorenz curve than to changes that occur in the upper part of the Lorenz curve. By contrast, the criterion of second-degree downward Lorenz dominance emphasizes changes that occur in the upper part of the Lorenz curve. Thus, an inequality

\(^{16}\) See Chateauneuf et al. [7] for an alternative proof of this result.
averse social decision-maker that employs the criterion of second-degree downward Lorenz dominance acts in favor of the Pigou-Dalton principle of transfers and the principle of first-degree UPTS. To characterize social preferences of these types we adopt the following definition.

**DEFINITION 2.6.** An inequality averse social decision-maker that supports the Pigou-Dalton principle of transfers and the principle of first-degree DPTS (UPTS) is said to exhibit *downside (upside) inequality aversion of first-degree*.

Theorems 2.2A and 2.2B demonstrate that the principles of upward and downward Lorenz dominance can be used to divide J_p-measures into wide families of inequality measures that differ in the measures’ sensitivity to changes (transfers) in the lower or upper part of the Lorenz curve. Members of the family \( \{ J_p : P \in P_{12}^* \} \) give more weight to changes that take place lower down in the Lorenz curve, whereas members of the family \( \{ J_p : P \in P_{22}^* \} \) give more weight to changes higher up in the Lorenz curve. Note that \( P(t) = 2t - t^2 \), the P-function that corresponds to the Gini coefficient, is neither included in \( P_{12}^* \) nor in \( P_{22}^* \). Since \( P''(t) = 0 \) for all \( t \), the Gini coefficient neither preserves second-degree upward Lorenz dominance nor second-degree downward Lorenz dominance apart from the case when the inequality in (i) of Theorems 2.2A and 2.2B holds strictly for \( u = 1 \) and \( u = 0 \), respectively.\(^{17}\) Thus, the suggestion of Muliere and Scarsini [33] that the Gini coefficient is coherent with second-degree upward Lorenz dominance requires a definition of second-degree that abandons the condition of strict inequality (for some \( u \in \{0,1\} \)). However, by assuming that the Lorenz curves cross only once the following results hold.\(^{18}\)

**PROPOSITION 2.1A.** Assume that \( L_1 \) and \( L_2 \) are singly intersecting Lorenz curves and \( L_1 \) crosses \( L_2 \) initially from above, and let \( G(L_1) \) and \( G(L_2) \) be the two corresponding Gini coefficients. Then the following statements are equivalent,

(i) \[ J_p(L_1) < J_p(L_2) \text{ for all } P \in P_{12}^* \]

(ii) \[ G(L_1) \leq G(L_2) \]

(Proof in Appendix)

\(^{17}\) Aaberge [1] gave an alternative interpretation of this property by demonstrating that the Gini coefficient attaches an equal weight to a given transfer irrespective of where it takes place in the income distribution, as long as the income transfer occurs between individuals with the same difference in ranks.

\(^{18}\) Zoli [49] provided a result similar to Proposition 2.1A for singly intersecting generalized Lorenz curves under the condition of equal means.
PROPOSITION 2.1B. Assume that $L_1$ and $L_2$ are singly intersecting Lorenz curves and $L_2$ crosses $L_1$ initially from above, and let $G(L_1)$ and $G(L_2)$ be the two corresponding Gini coefficients. Then the following statements are equivalent,

(i) $J_P(L_1) < J_P(L_2)$ for all $P \in P^*_2$

(ii) $G(L_1) \leq G(L_2)$.

The proof of Proposition 2.1B can be achieved by following the line of reasoning used in the proof of Proposition 2.1A. Note that Proposition 2.1A can be considered as a dual version of the results of Shorrocks and Foster [41] and Dardanoni and Lambert [10] that clarify the relationship between third-degree (upward) stochastic dominance, ordering conditions for the coefficient of variation and transfer sensitive measures of inequality.

As a preference ordering on $L$ the Gini coefficient in general favors neither the lower nor the upper part of the Lorenz curves. Therefore, if we restrict the ranking problem to Lorenz curves with equal Gini coefficients, second-degree upward and downward dominance coincide in the sense that a Lorenz curve $L_1$ that second-degree upward dominates a Lorenz curve $L_2$ is always second-degree downward dominated by $L_2$. Thus, it is clear that $L_1$ (and the corresponding distribution function) can be obtained from $L_2$ (the corresponding distribution function) by employing a set of Pigou-Dalton progressive transfers in combination with an equal set of Pigou-Dalton regressive transfers that leaves the Gini mean difference ($\mu G$) unchanged, and in which the progressive transfer of each pair of progressive/regressive transfers occurs lower down in the income distribution than the regressive transfer. We call such a change a downside mean-Gini-preserving transformation (downside MGPT)\(^{19}\). A formal definition of downside MGPT is given below.

DEFINITION 2.7A. Consider a discrete distribution $F$ and a Pigou-Dalton progressive transfer $\delta_i$ from a person with income $F^{-1}(t_i)$ to a person with income $F^{-1}(s_i)$ and a Pigou-Dalton regressive transfer $\gamma_i$ from a person with income $F^{-1}(u_i)$ to a person with income $F^{-1}(v_i)$ for $i = 1, 2, ..., N$. This sequence of combinations of progressive/regressive transfers is a downside mean-Gini-preserving transformation (downside MGPT) provided that $s_i < t_i < u_i < v_i$ for $i = 1, 2, ..., N$ and the $(\delta_i, \gamma_i)$ pairs are such that the Gini coefficient of the post-transfer distribution is equal to the Gini coefficient of $F$.

\(^{19}\) Note that a downside MGPT is equivalent to the favorable composite positional transfer discussed by Zoli [50].
Thus, by applying the downside MGPT inequality is transferred from lower to higher parts of the Lorenz curve. By contrast, a social decision-maker who favors second-degree downward Lorenz dominance will apply the progressive transfer of each pair of progressive/regressive transfers higher up in the income distribution than the regressive transfer. Such a change will be called an upside mean-Gini-preserving transformation (upside MGPT). In this case inequality is transferred from the higher to the lower parts of the Lorenz curve and the corresponding income distribution.

**DEFINITION 2.7B.** Consider a discrete distribution \( F \) and a Pigou-Dalton regressive transfer \( \delta_i \) from a person with income \( F^{-1}(s_i) \) to a person with income \( F^{-1}(t_i) \) and a Pigou-Dalton progressive transfer \( \gamma_i \) from a person with income \( F^{-1}(v_i) \) to a person with income \( F^{-1}(u_i) \) for \( i = 1, 2, \ldots, N \). This sequence of combinations of regressive/progressive transfers is an upside mean-Gini-preserving transformation (upside MGPT) provided that \( s_i < t_i < u_i < v_i \) for \( i = 1, 2, \ldots, N \) and the \((\delta_i, \gamma_i)\) pairs are such that the Gini coefficient of the post-transfer distribution is equal to the Gini coefficient of \( F \).

The following theorem demonstrates that a Lorenz curve \( L_1 \) that second-degree upward dominates a Lorenz curve \( L_2 \) can be obtained from \( L_2 \) be a sequence of downside MGPTs’ and that the Lorenz curve \( L_2 \) can be obtained from \( L_1 \) by a sequence of upside MGPTs’.

**THEOREM 2.3.** Let \( L_1 \) and \( L_2 \) be Lorenz curves with equal Gini coefficients. Then the following statements are equivalent,

(i) \( L_1 \) second-degree upward dominates \( L_2 \)

(ii) \( L_2 \) second-degree downward dominates \( L_1 \)

(iii) \( L_1 \) can be obtained from \( L_2 \) by a sequence of downside MGPTs’.

(iv) \( L_2 \) can be obtained from \( L_1 \) by a sequence of upside MGPTs’.

(Proof in Appendix)

Note that the MGPT transformations are analogous to the mean-variance-preserving transformation (MVPT) introduced by Menezes et al. [32] and used by Shorrocks and Foster [41], Dardanoni and Lambert [10] and Davis and Hoy [12] as a basis for analysing the implications of third-degree stochastic dominance on measurement of inequality and social welfare. The major difference between these methods of transformation is that the (downside) MGPT is equivalent to second-degree upward Lorenz dominance (third-degree upward inverse stochastic dominance), whereas the MVPT is equivalent to third degree (upward) stochastic dominance. Moreover, the MGPT relies on the Gini mean difference rather than the variance as a measure of spread. Note that the Gini mean difference was used as an (robust) alternative to the variance as a measure of spread long before Gini introduced it as a measure of inequality (see David [11]).

An alternative proof of the equivalence between (i) and (iii) was provided by Zoli [50].
When the ranking problem is restricted to Lorenz curves with equal Gini coefficients then the condition that a downside MGPT (an upside MGPT) reduces inequality is equivalent to the condition for satisfying the principle of first-degree DPTS (UPTS).

3. LORENZ DOMINANCE OF I

Since situations where second-degree (upward or downward) Lorenz dominance does not provide unambiguous ranking of Lorenz curves may arise, it will be useful to introduce weaker dominance criteria than second-degree Lorenz dominance. To this end we will introduce two hierarchical sequences of nested Lorenz dominance criteria; one departs from second-degree upward Lorenz dominance and the other from second-degree downward Lorenz dominance. As explained in Section 2, the choice between second-degree upward and downward Lorenz dominance clarifies whether focus is turned to changes that take place in the lower or upper part of the income distribution. Thus, a person who favors second-degree upward Lorenz dominance would most likely prefer third-degree and higher degrees of upward Lorenz dominance to third-degree and higher degrees of downward Lorenz dominance. Conversely, when the value judgment of a person is consistent with the criterion of second-degree downward Lorenz dominance, higher degrees of downward Lorenz dominance are likely more acceptable than higher degrees of upward Lorenz dominance.

As will become evident below it is convenient to use the following notation,

\[
L_i^2(u) = \int_0^u L(t) \, dt = \frac{1}{\mu} \int_0^u (u - t)^i F^{-i}(t) \, dt, \quad 0 \leq u \leq 1, \quad (5)
\]

\[
L_i^3(u) = \int_0^u L_i^2(t) \, dt, \quad 0 \leq u \leq 1, \quad i = 2, 3, \ldots,
\]

and

\[
\tilde{L}_i^2(u) = \int_u^1 (1 - L(t)) \, dt = \frac{1}{\mu} \int_u^1 (t - u)^i F^{-i}(t) \, dt, \quad 0 \leq u \leq 1, \quad (6)
\]

\[
\tilde{L}_i^3(u) = \int_u^1 \tilde{L}_i^2(t) \, dt, \quad 0 \leq u \leq 1, \quad i = 2, 3, \ldots.
\]

Now, using integration by parts, we obtain the following alternative expressions for \(L_i^{i+1}\) and \(\tilde{L}_i^{i+1}\), respectively,

\[
L_i^{i+1}(u) = \frac{1}{(i - 1)!} \int_0^u (u - t)^{i+1} L(t) \, dt = \frac{1}{i!\mu} \int_0^u (u - t)^i F^{-i}(t) \, dt, \quad i = 2, 3, \ldots \quad (7)
\]
and

\[
L_i^{i+1}(u) = \frac{1}{(i-1)!} \int_u^1 (t-u)^{i+1} (1-L(t))dt = \frac{1}{i! \mu} \int_u^1 (t-u)^i F^{-1}(t)dt, \quad i = 2, 3, \ldots \tag{8}
\]

It is easily verified that \(L_i^{i+1}(1)\) defined by (7) is a linear transformation of a measure of inequality that belongs to the extended Gini family of inequality measures\(^{22}\) \(\{G_i : i \geq 1\}\),

\[
L_i^{i+1}(1) = \frac{1}{(i+1)!} (1-G_i(L)), \quad i = 1, 2, \ldots \tag{9}
\]

where

\[
G_i(L) = 1 - i(i+1) \int_0^1 (1-u)^{i+1} L(u)du = \frac{1}{\mu} \int_0^x (1-F(x))\left(1-(1-F(x))^i\right)dx, \quad i \geq 1. \tag{10}
\]

Moreover, from the definition (8) of \(L\) we get that

\[
L_i^{i+1}(0) = \frac{1}{(i+1)!} (iD_i(L) + 1), \quad i = 1, 2, \ldots \tag{11}
\]

where

\[
D_i(L) = 1 - (i+1) \int_0^1 u^{i+1} L(u)du = \frac{1}{\mu} \int_0^x F(x)\left(1-F(x)^i\right)dx, \quad i = 1, 2, \ldots \tag{12}
\]

and \(\{D_i : i = 1, 2, \ldots\}\) is an alternative “generalized” Gini family of inequality measures denoted the Lorenz family of inequality measures\(^{22}\), where \(D_1 = G_1\) is equal to the Gini coefficient.

As was demonstrated by Aaberge [1] there is a one-to-one correspondence between subfamilies of the extended Gini and the Lorenz families of inequality measures shown by the following equation

\[
G_i(L) = 1+(i+1) \sum_{k=1}^i (-1)^k \binom{i}{k} \frac{k}{k+1} (1-D_k(L)), \quad i = 1, 2, \ldots \tag{13}
\]

Thus, the extended Gini subfamily \(\{G_i(L) : i = 1, 2, \ldots, r\}\) is uniquely determined by the corresponding Lorenz subfamily \(\{D_i(L) : i = 1, 2, \ldots, r\}\) for any integer \(r\).

\(^{22}\) The extended Gini family of inequality measures was introduced by Donaldson and Weymark [14, 15] and Yitzhaki [48].
Expressions (7) and (8) show that $L_{i+1}$ places more weight on changes in the lower and $\bar{L}_{i+1}$ on changes in the upper part of the Lorenz curve as $i$ increases.

As generalizations of Definitions 2.4A and 2.4B we introduce the notions of $i^{th}$-degree upward and downward Lorenz dominance\(^{24}\). Note that subscripts $i$ and $j$ in the notation $L_i$ and $\bar{L}_i$ used below refer to dominance of $i^{th}$-degree for Lorenz curve $L_j$ and that $L_j$ is the Lorenz curve $L_j$ and $\bar{L}_j = 1 - L_j$.

**DEFINITION 3.1A.** A Lorenz curve $L_1$ is said to $i^{th}$-degree upward dominate a Lorenz curve $L_2$ if

$$L_i(u) \geq L_j(u) \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \{0,1\}$.

**DEFINITION 3.1B.** A Lorenz curve $L_1$ is said to $i^{th}$-degree downward dominate a Lorenz curve $L_2$ if

$$\bar{L}_i(u) \geq \bar{L}_j(u) \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \{0,1\}$.

Note that $(i + 1)^{th}$-degree upward and downward Lorenz dominance are less restrictive dominance criteria than $i^{th}$-degree upward and downward Lorenz dominance and thus can prove to be useful decision criteria in situations where $i^{th}$-degree dominance does not yield an unambiguous ranking of Lorenz curves.

It follows from the definitions (7) and (8) of $L_i$ and $\bar{L}_i$ that

$$L_1(u) \geq L_2(u) \text{ for all } u \text{ implies } L_{i+1}(u) \geq \bar{L}_{i+1}(u) \text{ for all } u,$$

and that

$$\bar{L}_2(u) \geq \bar{L}_1(u) \text{ for all } u \text{ implies } \bar{L}_{i+1}(u) \geq \bar{L}_{i+1}(u) \text{ for all } u,$$

---

\(^{23}\) The Lorenz family of inequality measures was introduced by Aaberge [1] and proves to be a subclass of the "illfare-ranked single-series Gini" discussed by Donaldson and Weymark [14] and Bossert [6].

\(^{24}\) A similar definition of $i^{th}$ degree (upward) inverse stochastic dominance was introduced by Muliere and Scarsini [33]. Note that Definitions 3.1A and 3.1B do not require any restrictions on the Lorenz curves (or the distribution functions) and thus differ in this sense from the definitions of stochastic dominance proposed by Whitmore [45] and Chew [8].
which means that \((i+1)^{th}\) -degree upward Lorenz dominance preserves \(i^{th}\)-degree upward Lorenz dominance and that \((i+1)^{th}\) -degree downward Lorenz dominance preserves \(i^{th}\)-degree downward Lorenz dominance.

Thus, the various degrees of upward and downward Lorenz dominance form two separate sequences of nested dominance criteria, which turn out to be useful for dividing the \(J_P\)-family of inequality measures into nested subfamilies. To this end it will be convenient to introduce the following notation. Let \(P^{(j)}\) denote the \(j^{th}\) derivative of \(P\) and let \(P_{ii}, P_{ii}^*, P_{ii}^{**}, P_{ii}^{***}, P_{2i}, P_{2i}^*, P_{2i}^{**}, P_{2i}^{***}\) be families of preference functions defined by

\[
P_{ii} = \left\{ P : P \in P_i, P^{(j)} \text{ is continuous on } [0, 1] \text{ and } P^{(j)}(1) = 0, j = 2, 3, \ldots, i \right\},
\]

\[
P_{ii}^* = \left\{ P : P \in P_{ii} \text{ and } (-1)^j P^{(j-1)}(t) > 0 \text{ for } t \in (0, 1), j = 1, 2, \ldots, i \right\},
\]

\[
P_{ii}^{**} = \left\{ P : P \in P_{ii} \text{ and } (-1)^j P^{(j-1)}(t) > 0 \text{ for } t \in (0, 1) \right\},
\]

\[
P_{ii}^{***} = \left\{ P : P \in P_i, P^{(j)} \text{ is continuous on } [0, 1], (-1)^j P^{(j-1)}(t) > 0 \text{ for } t \in (0, 1) \text{ and } (-1)^j P^{(j)}(1) \geq 0, j = 2, 3, \ldots, i \right\},
\]

\[
P_{2i} = \left\{ P : P \in P_i, P^{(j)} \text{ is continuous on } [0, 1] \text{ and } P^{(j)}(0) = 0, j = 2, 3, \ldots, i \right\},
\]

\[
P_{2i}^* = \left\{ P : P \in P_{2i} \text{ and } P^{(j-1)}(t) < 0 \text{ for } t \in (0, 1), j = 1, 2, \ldots, i \right\},
\]

\[
P_{2i}^{**} = \left\{ P : P \in P_{2i} \text{ and } P^{(j-1)}(t) < 0 \text{ for } t \in (0, 1) \right\},
\]

and

\[
P_{2i}^{***} = \left\{ P : P \in P_i, P^{(j)} \text{ is continuous on } [0, 1], P^{(j-1)}(t) < 0 \text{ for } t \in (0, 1) \text{ and } P^{(j)}(0) \leq 0, j = 2, 3, \ldots, i \right\},
\]

respectively. Note that \(P_{ii}^* \subset P_{ii}^{**} \subset P_{ii}^{***}\) and \(P_{2i}^* \subset P_{2i}^{**} \subset P_{2i}^{***}\).

The subfamilies of the \(J_P\)-family formed by \(P_{ii}^{**}\) and \(P_{2i}^{**}\) are characterized by the following theorems.

**Theorem 3.1A.** Let \(L_1\) and \(L_2\) be members of \(L\). Then the following statements are equivalent.

(i) \(L_1\) \(i^{th}\)-degree upward dominates \(L_2\).
(ii) \( J_\alpha(L_1) < J_\alpha(L_2) \) for all \( \alpha \in P_{11}^- \).

(Proof in Appendix).

THEOREM 3.1B. Let \( L_1 \) and \( L_2 \) be members of \( L \). Then the following statements are equivalent,

(i) \( L_1 \) \( \text{th} \)-degree downward dominates \( L_2 \)

(ii) \( J_\alpha(L_1) < J_\alpha(L_2) \) for all \( \alpha \in P_{11}^- \).

(Proof in Appendix).

The criteria of Lorenz dominance offer convenient computational methods for applied work. As is demonstrated by Theorems 3.1A and 3.1B this approach is particular attractive since it provides identification of the restrictions on the preference function \( \alpha \) that are needed to reach unambiguous rankings of Lorenz curves.\(^{25}\) As will be demonstrated in Section 4 the extended Gini inequality measure \( G_k \) satisfies the conditions \( P^j(1) = 0, j = 2, 3, \ldots, k \), whereas the Lorenz family measure \( D_k \) satisfies the conditions \( P^j(0) = 0, j = 2, 3, \ldots, k \).

To judge the normative significance of \( \beta \)-degree upward and downward Lorenz dominance, it appears helpful to strengthen the principles of first-degree downside and upside positional transfer sensitivity to be more sensitive to transfers that take place lower down (higher up) in the income distribution.\(^{26}\) To this end it will be useful to introduce the following notation. Let \( \Delta I_{1,1}(\delta, h_1, h_2) \) be defined by

\[
\Delta I_{1,1}(\delta, h_1, h_2) = \Delta I_{1,1}(\delta, h_1) - \Delta I_{1,1}(\delta + h_2, \delta + h_3)
\]

(14)

where \( \Delta I_{1,1}(\delta, h) \) is defined by (4).

DEFINITION 3.3A. Consider a discrete income distribution \( F \), an inequality measure \( I \) that obeys the first-degree DPTS, and rank-preserving transfers \( \delta \) from individuals with ranks \( s + h_1, s + h_1 + h_2, t + h_1, \) and \( t + h_1 + h_2 \) to individuals with ranks \( s, s + h_1, t \) and \( t + h_2 \) in \( F \). Then \( I \) is said to

\(^{25}\) Note that Muliere and Scarsini [33] provided a characterization of \( \beta \)-degree upward Lorenz dominance (inverse stochastic dominance) in terms of order conditions for a subfamily of \( P_{11}^- \).

\(^{26}\) Note that Fishburn and Willig [19] introduced an extension of Kolm’s principle of diminishing transfers to higher-order transfer principles and demonstrated that these principles are associated to higher orders of upward stochastic dominance.
satisfy the principle of second-degree downside positional transfer sensitivity, the second-degree DPTS, if

\[ \Delta_2 I_{s,t} (\delta, h_1, h_2) > 0 \text{ when } s < t. \]

DEFINITION 3.3B. Consider a discrete income distribution F, an inequality measure I that obeys the first-degree UPTS, and rank-preserving transfers \( \delta \) from individuals with ranks \( s + h_1, s + h_1 + h_2, t + h_1, \) and \( t + h_1 + h_2 \) to individuals with ranks \( s, s + h_1, t \) and \( t + h_2 \) in F. Then I is said to satisfy the principle of second-degree upside positional transfer sensitivity, the second-degree UPTS, if

\[ \Delta_2 I_{s,t} (\delta, h_1, h_2) > 0 \text{ when } s < t. \]

Note that \( \Delta_1 I_{s,t} < 0 \) for any \( s < t \) when I obeys the first-degree UPTS. Since the principle of second-degree UPTS is meant to strengthen the principle of first-degree UPTS, it follows from (14) that this is obtained when \( \Delta_2 I_{s,t} > 0 \) for \( s < t \). Thus, we can only discern between second-degree DPTS and second-degree UPTS if these principles are required to be linked to first-degree DPTS and first-degree UPTS, respectively. Thus, when a sequence of first-degree DPTS (UPTS) transfers is valued more the lower down (higher up) the sequence of the transfers occurs, the sequence of transfers is made in line with the principle of second-degree downside (upside) positional transfer sensitivity.

To deal with \( i^{th} \)-degree Lorenz dominance it is convenient to introduce the notation \( \Delta_1 I_{s,t} (\delta, h_1, h_2, ..., h_i) \) defined by

\[
\Delta_1 I_{s,t} (\delta, h_1, h_2, ..., h_i) = \Delta_i I_{s,t} (\delta, h_1, h_2, ..., h_{i-1}) - \Delta_{i+1} I_{s+h_i + s, h_i} (\delta, h_1, h_2, ..., h_{i-1}), \quad i = 3, 4, \ldots
\]  

DEFINITION 3.4A. Consider a discrete income distribution F, an inequality measure I that obeys the \((i - 1)^{th}\)-degree DPTS, and rank-preserving transfers \( \delta \) from individuals with ranks \( s + h_1, s + h_1 + h_2, ..., s + h_1 + h_i, ..., s + h_1 + h_2 + ... + h_i, t + h_1, t + h_1 + h_2, ..., t + h_1 + h_i, ..., t + h_1 + h_2 + ... + h_i \) to individuals with ranks \( s, s + h_2, ..., s + h_i, ..., s + h_1 + h_2 + ... + h_i, t, t + h_2, ..., t + h_i, ..., t + h_1 + h_2 + ... + h_i \) in F. Then I is said to satisfy the principle of \( i^{th} \)-degree downside positional transfer sensitivity, the \( i^{th} \)-degree DPTS, if

\[ \Delta I_{s,t} (\delta, h_1, h_2, ..., h_i) > 0 \text{ when } s < t. \]
DEFINITION 3.4B. Consider a discrete income distribution $F$, an inequality measure $I$ that obeys the $(i-1)^{th}$-degree UPTS, and rank-preserving transfers $\delta$ from individuals with ranks $s+h_i$, $s+h_i+h_{i-1} + \ldots + s$, $s$, $s+h_i+h_{i-1} + \ldots + s$, to individuals with ranks $s+h_i+h_{i-1} + \ldots + s$, $s+h_i+h_{i-1} + \ldots + s$, $s+h_i+h_{i-1} + \ldots + s$, $s+h_i+h_{i-1} + \ldots + s$, $s+h_i+h_{i-1} + \ldots + s$, $s$. Then $I$ is said to satisfy the principle of $i^{th}$-degree upside positional transfer sensitivity, the $i^{th}$-degree UPTS, if

$$\Delta I_{s,t}(\delta, h_i, h_{i-1}, \ldots, h_1) < 0 \text{ when } s < t \text{ and } i = 2k - 1, k = 1,2,\ldots,$$

and

$$\Delta I_{s,t}(\delta, h_i, h_{i-1}, \ldots, h_1) > 0 \text{ when } s < t \text{ and } i = 2k, k = 1,2,\ldots.$$
(iii) \( J_p(L_1) < J_p(L_2) \) for all \( P \in P_i \), being such that \( J_p \) obeys the Pigou-Dalton principle of transfers and the principles of UPTS up to and including \((i-1)^{th}\)-degree.

The proof of Theorem 3.2B can be achieved by following the line of reasoning used in the proof of Theorem 3.2A.

As noted above the motivation for introducing the principles of DPTS (UPTS) was to successively strengthen the emphasis of transfers taking place lower down (higher up) in the income distribution. To characterize social preferences that are consistent with these principles we adopt the following definition.

**DEFINITION 3.5.** A social decision-maker that supports the Pigou-Dalton principle of transfers and the principles of DPTS (UPTS) up to and including \( i^{th} \)-degree is said to exhibit **downside (upside) inequality aversion of \( i^{th} \)-degree**.

By adding the condition of dominating extended Gini coefficients \( G_k \) for \( k = 1, 2, ..., i - 1 \) to the condition of \( i^{th} \)-degree upward Lorenz dominance it follows from the proof of Theorem 3.1A that the conditions \( P^{(j)}(1) = 0, j = 2, 3, ..., i \) can be replaced by less restrictive conditions for \( P \), which means that the subfamily of \( J_p \)-measures that preserves a “restricted” \( i^{th} \)-degree upward Lorenz dominance condition is larger than the subfamily of \( J_p \)-measures that preserves \( i^{th} \)-degree upward Lorenz dominance. Moreover, as indicated above, the latter is a subfamily of the former.

**THEOREM 3.3A.** Let \( L_1 \) and \( L_2 \) be members of \( L \). Then the following statements are equivalent,

(i) \( L_1 \) \( i^{th} \)-degree upward dominates \( L_2 \) and \( G_k(L_j) \leq G_k(L_2) \) for \( k = 1, 2, ..., i - 1 \)

(ii) \( J_p(L_1) < J_p(L_2) \) for all \( P \in P_{ii}^{++} \).

(Proof in Appendix.)

**THEOREM 3.3B.** Let \( L_1 \) and \( L_2 \) be members of \( L \). Then the following statements are equivalent,

(i) \( L_1 \) \( i^{th} \)-degree downward dominates \( L_2 \) and \( D_k(L_1) \leq D_k(L_2) \) for \( k = 1, 2, ..., i - 1 \)

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27 Wang and Young [43] provide a result similar to Theorem 3.3B for intersecting distribution functions. However, their result relies on the condition of negative derivatives (up to order \( i \)) of \( P \) whereas the condition for \( P \) used in Theorem 3.3B is less strict. Moreover, Wang and Young [43] don’t appear to be aware of the fact that their result concerns downward rather than upward inverse stochastic dominance.
(ii) $J_p(L_1) < J_p(L_2)$ for all $P \in P_{2,i}^{**}$.

(Proof in Appendix.)

As we demonstrated in Section 2, first-degree DPTS (UPTS) could be given an alternative interpretation in terms of a mean-Gini-preserving transformation. This equivalence arises due to the fact that second-degree upward and downward Lorenz dominance “coincide” when the Gini coefficient (or the mean and the Gini mean difference) of the Lorenz curves are equal. By assuming that $L^3(1)$ defined by (9) is kept fixed, a similar interpretation of second-degree DPTS in terms of a mean-Gini-$L^3(1)$-preserving transformation can be obtained. Requiring $L^3(1)$ to be equal across Lorenz curves is equivalent to require that the extended Gini measure of inequality $G_2(L)$ (or its absolute version and the mean) is kept fixed. Thus, in order to obtain the upward dominating $L^2$-curve from the dominated $L^2$-curve we may use combinations of downside and upside MGPTs’ that are carried out under the conditions of equal Gini coefficients and equal $G_2$-coefficients. Similarly, we find that the downward dominating $L^2$-curve can be obtained from the dominated $L^2$-curve by combinations of downside and upside MGPTs’ provided that $\tilde{L}(0)$ is kept fixed, which according to (11) is equivalent to require that the $D_2$-coefficients are equal. However, when the requirement of equal $D_2$-coefficients are made in combination with the condition of equal Gini-coefficients, it follows from equation (13) that this is equivalent to require equal $G_2$-coefficients in combination with equal Gini coefficients. Formal definitions of downside and upside mean-Gini-$G_2$-preserving (or mean-Gini-$L^3(1)$-preserving) transformations are given by

**DEFINITION 3.6.** Consider a discrete distribution $F$, a downside MGPT of two amounts $(\delta_1, \gamma_1)$ and an upside MGPT of two amounts $(\delta_2, \gamma_2)$. This pair of transformations is a downside mean-Gini-$G_2$-preserving transformation (downside MG$_2$PT) provided that the pairs $(\delta_1, \gamma_1)$ and $(\delta_2, \gamma_2)$ are such that the $G_2$-coefficient of the post-transfer distribution is equal to the $G_2$-coefficient of $F$, and the downside MGPT occurs lower down in the income distribution than the upside MGPT. By contrast, when the upside MGPT occurs lower than the downside MGPT we say that the pair of transformations is an upside mean-Gini-$G_2$-preserving transformation (upside MG$_2$PT).

As indicated above there is a similarity between third-degree and second-degree Lorenz dominance in the sense that both dominance criteria can be given a normative characterization in terms of principles for positional transfer sensitivity and mean-“spread”-preserving transformations. Moreover, as for the second-degree dominance criteria, upward and downward third-degree Lorenz dominance require the same measures of “spread” (inequality) to be kept fixed in order to obey the
principle of mean-“spread”-preserving transformation. Keeping the Gini coefficient and G$_2$ fixed is, as noted above, equivalent to keeping the Gini coefficient and D$_2$ fixed. This means that upside as well as downside MG$_2$PT relies on fixed G$_2$- and D$_2$-coefficients. Observe that the scaled-up (the absolute) version of D$_2$ (or $\hat{L}^i(0)$) can be considered as a measure of right spread, i.e. it is more sensitive to changes that affect the spread in the upper part of the income distribution rather than in changes that affect the spread in the lower part of the income distribution. By contrast, the scaled-up version of G$_2$ can be considered as a measure of left-spread. Accordingly, it is more sensitive to changes in spread that occur in the lower rather in the upper part of the income distribution. Thus, upside and downside MG$_2$PT place more emphasis on the tails than the upside and downside mean-Gini-preserving transformation. The downside mean-Gini-G$_2$-preserving transformation reduces inequality (spread) in the lower part of the income distribution at the expense of increased inequality in the middle/upper part of the income distribution, whereas the upside mean-Gini-G$_2$-preserving transformation reduces inequality (spread) in the upper part of the income distribution at the expense of increased inequality in the middle/lower part of the income distribution. In order to place even stronger emphasis on the tails we may introduce higher-order mean-“spread”-preserving transformations.

DEFINITION 3.7. Consider a discrete distribution F, a downside MG$_{i-1}$PT of two amounts $(\delta_1, \gamma_1)$ and an upside MG$_{i-1}$PT of two amounts $(\delta_2, \gamma_2)$. This pair of transformations is a **downside mean-Gini-G$_2$-…-G$_i$-preserving transformation (downside MG$_i$PT)** provided that the pairs $(\delta_1, \gamma_1)$ and $(\delta_2, \gamma_2)$ are such that the G$_r$-coefficient of the post-transfer distribution is equal to the G$_r$-coefficient of F, and the downside MG$_{i-1}$PT occurs lower down in the income distribution than the upside MG$_{i-1}$PT. By contrast, when the upside MG$_{i-1}$PT occurs lower than the downside MG$_{i-1}$PT we say that the pair of transformations is an **upside mean-Gini-G$_2$-…-G$_i$-preserving transformation (upside MG$_i$PT)**.

The following theorems demonstrate that a Lorenz curve L$_1$ that $(i+1)^{\text{th}}$-degree upward (downward) dominates a Lorenz curve L$_2$ can be obtained from L$_2$ by a sequence of downside (upside) MG$_i$PTs’.

THEOREM 3.4A. Let L$_1$ and L$_2$ be Lorenz curves with equal Gini coefficients and $G_j(L_1) = G_j(L_2)$, $j = 2, 3, ..., i$. Then the following statements are equivalent,

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28 In contrast to the absolute Gini-coefficient note that the absolute versions of G$_i$ defined by (10) and D$_i$ defined by (12) do not fulfill the standard conditions of being measures of spread (dispersion) when $i>1$, but can be used as measures of left and right spread, respectively. See Fernández-Ponce et al. [17] and Shaked and Shanthikumar [39] for discussions on measurement of right spread.
Theorem 3.4B. Let $L_1$ and $L_2$ be Lorenz curves with equal Gini coefficients and $G_j(L_i) = G_j(L_2)$, $j = 2, 3, \ldots, i$. Then the following statements are equivalent.

(i) $L_1$ $(i+1)^{th}$-degree upward dominates $L_2$

(ii) $J_p(L_1) < J_p(L_2)$ for all $P$ such that $P^{(i+2)}$ is continuous on $[0,1]$ and $(-1)^{i+1} P^{(i+2)}(u) > 0$ for $u \in (0,1)$

(iii) $L_1$ can be obtained from $L_2$ by a sequence of upside MGPTs'.

(Proof in Appendix.)

Theorem 3.4B. Let $L_1$ and $L_2$ be Lorenz curves with equal Gini coefficients and $G_j(L_i) = G_j(L_2)$, $j = 2, 3, \ldots, i$. Then the following statements are equivalent.

(i) $L_1$ $(i+1)^{th}$-degree downward dominates $L_2$

(ii) $J_p(L_1) < J_p(L_2)$ for all $P$ such that $P^{(i+2)}$ is continuous on $[0,1]$ and $P^{(i+2)}(u) < 0$ for $u \in (0,1)$

(iii) $L_1$ can be obtained from $L_2$ by a sequence of upside MGPTs'.

The proof of Theorem 3.4B can be constructed by following exactly the line of reasoning used in the proof of Theorem 3.4A.

Since downside as well as upside MGPT relies on equal Gini coefficients and $G_j(L_i) = G_j(L_2)$, $j = 2, 3, \ldots, i$, it will be of interest to explore the relationship between these two transformation approaches. To this end we draw on Theorems 3.4A and 3.4B, from which we can derive the following results.

Corollary 3.1A. Let $L_1$ and $L_2$ be Lorenz curves with equal Gini coefficients and $G_j(L_i) = G_j(L_2)$, $j = 2, 3, \ldots, i$. Then the following statements are equivalent for $i = 2k, k = 1, 2, \ldots$.

(i) $L_1$ $(i+1)^{th}$-degree upward dominates $L_2$

(ii) $L_1$ $(i+1)^{th}$-degree downward dominates $L_2$

(iii) $J_p(L_1) < J_p(L_2)$ when $P^{(i+2)}(u) < 0$ for $u \in (0,1)$

(iv) $L_1$ can be obtained from $L_2$ by a sequence of downside MGPTs'

(v) $L_1$ can be obtained from $L_2$ by a sequence of upside MGPTs'.

Corollary 3.1B. Let $L_1$ and $L_2$ be Lorenz curves with equal Gini coefficients and $G_j(L_i) = G_j(L_2)$, $j = 2, 3, \ldots, i$. Then the following statements are equivalent for $i = 2k - 1, k = 1, 2, \ldots$. 

25
(i) \( L_1 \ (i + 1)^{th} \)-degree upward dominates \( L_2 \)

(ii) \( L_2 \ (i + 1)^{th} \)-degree downward dominates \( L_1 \)

(iii) \( J_p(L_1) < J_p(L_2) \) when \( p^{(i+2)}(u) > 0 \) for \( u \in (0,1) \)

(iv) \( L_1 \) can be obtained from \( L_2 \) by a sequence of downside MGPTs’

(v) \( L_2 \) can be obtained from \( L_1 \) by a sequence of upside MGPTs’.

**REMARK.** As demonstrated above the condition of equal Gini coefficients and \( G_j(L_1) = G_j(L_2), j = 2,3,...,i \) is equivalent to the condition of equal Gini coefficients and \( D_j(L_1) = D_j(L_2), j = 2,3,...,i \). Thus, the downside and upside mean-Gini-G_2-…-G_\(i\)-preserving transformations could alternatively have been denoted the downside and upside mean-Gini-D_2-…-D_\(i\)-preserving transformations.

The proposed sequences of dominance criteria along with the results of Theorems 3.1A-3.4A and 3.1B-3.4B suggest two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered by successively narrowing the class of inequality measures under consideration. As the dominance criteria of each sequence are nested these strategies also allow us to identify the value judgments that are needed to reach an unambiguous ranking of Lorenz curves. It follows from Theorem 3.2A that \( J_p \)-measures derived from P-functions with derivatives between second and \( i^{th} \) order that alternate in sign \( (-1)^{j-1} p^{(j)}(t) > 0, j = 2,3,...,i \) preserve upward Lorenz dominance of all degrees lower than and equal to \( i-1 \). Thus, as demonstrated by Theorem 3.2A their sensitivity to changes that occur in the lower part of the income distribution (and the Lorenz curve) increases as \( i \) increases. By contrast, Theorem 3.2B shows that \( J_p \)-measures derived from P-functions with negative derivatives of order two and up to \( i \) \( p^{(j)}(t) < 0, j = 2,3,...,i \) preserve downward Lorenz dominance of all degrees lower than and equal to \( i-1 \). Theorem 3.2B demonstrates that this means that they increase their sensitivity to changes that occur in the upper part of the Lorenz curve as \( i \) increases. Note that the theorems, propositions and corollaries introduced above are only valid for finite \( i \). At the extreme, as \( i \rightarrow \infty \), observe that

\[
(i + 1)! L^{(i+1)}(u) \rightarrow \begin{cases} 0, & 0 \leq u < 1 \\ \frac{F^{-1}(0+)}{u}, & u = 1 \end{cases}
\]

(16)

and
\[(i+1)!\tilde{L}_{i+1}(u) \rightarrow \begin{cases} 
\frac{F^{-1}(1)}{\mu}, & u = 0 \\
0, & 0 < u \leq 1, 
\end{cases} \quad (17)\]

where \(F^{-1}(0^+)\) and \(F^{-1}(1)\) denote the lowest and highest income, respectively. Hence, at the limit upward and downward Lorenz dominance solely depend on the income share of the worst-off and best-off income recipient, respectively. At the extreme upward Lorenz dominance is solely concerned with transfers that benefit the poorest unit. By contrast, downward Lorenz dominance solely focuses on transferring money from the richest to anyone else.

**REMARK.** Restricting the comparisons of Lorenz curves to distributions with equal means the various dominance results of Sections 2 and 3 are valid for generalized Lorenz curves and also apply to the so-called dual theory representation for choice under uncertainty introduced by Yaari [46, 47].

4. THE RELATIONSHIP BETWEEN DOWNWARD AND UPWARD LORENZ DOMINANCE AND GENERALIZED GINI FAMILIES OF INEQUALITY MEASURES

The dominance results in Sections 2 and 3 show that application of the criteria of upward Lorenz dominance requires a higher degree of aversion to downside inequality the higher is the degree of upward Lorenz dominance. A similar relationship holds between downward Lorenz dominance and aversion to upside inequality aversion. As suggested in Section 3 the highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the worst-off income recipient. Thus, the most downside inequality averse \(J_P\)-measure that is obtained as the preference function approaches

\[P_d(t) = \begin{cases} 
0, & t = 0 \\
1, & 0 < t \leq 1, 
\end{cases} \quad (18)\]

can be considered as the \(J_P\)-measure that exhibits the highest degree of downside inequality aversion. As \(P_d\) is not differentiable, it is not a member of the family \(P_1\) of inequality averse preference functions, but it is recognizable as the upper limit of inequality aversion for members of \(P_1\). Inserting (18) in (2) yields

\[J_{P_d}(L) = 1 - \frac{F^{-1}(0^+)}{\mu}, \quad (19)\]

Hence, the inequality measure \(J_{P_d}\) corresponds to the Rawlsian maximin criterion. Since \(J_{P_d}\) is compatible with the limiting case of upward Lorenz dominance the Rawlsian (relative) maximin criterion preserves all degrees of upward Lorenz dominance and rejects downward Lorenz dominance.
By contrast, the \( J_P \)-measure that is obtained as \( P \) approaches
\[
P_u(t) = \begin{cases} 
1, & 0 \leq t < 1 \\
0, & t = 1,
\end{cases}
\] (20)

exhibits the highest degree of upside inequality aversion. Inserting (20) in (2) yields
\[
J_{P_u}(L) = 1 + \frac{F_1^{-1}(l)}{\mu}.
\] (21)

Thus, \( J_{P_u} \), which we will denote the relative minimax criterion, is “dual” to the Rawlsian (relative) maximin criterion in the sense that it is compatible with the limiting case of downward Lorenz dominance. When the comparison of Lorenz curves is based on the relative minimax criterion the Lorenz curve for which the largest relative income is smaller is preferred, regardless of all other differences. The only transfer which decreases inequality is a transfer from the richest unit to anyone else.

Based on the results in Sections 2 and 3, we shall now demonstrate how the above Lorenz dominance results can be applied to evaluate the ranking properties of the Lorenz and the extended Gini families of inequality measures. The extended Gini family is defined by (10). Note that \( \{G_k : k > 0\} \) is a subfamily of \( \{J_P : P \in P_1\} \) formed by the following family of \( P \)-functions,
\[
P_{ik}(t) = 1 - (1 - t)^{k+1}, \ k \geq 0.
\] (22)

Differentiating \( P_{ik} \) defined by (22), we find that
\[
P_{ik}^{(j)}(t) = \begin{cases} 
(-1)^{j+1} \frac{(k+1)!}{(k-j+1)!} (1-t)^{k-j+1}, & j = 1, 2, \ldots, k+1 \\
0, & j = k+2, k+3, \ldots
\end{cases}
\] (23)

\[29 \text{ Note that the normalization condition } P'(0)=1 \text{ is ignored in this case.} \]
As can be observed from (23) the weight function $P_{1k}$ of the extended Gini family of inequality measures is a member of $P_i^*$ (and $P_{i}^{**}$) for $k=i,i+1,...$. Thus, we get the following result from Theorem 3.2A.\textsuperscript{30}

\begin{equation*}
\text{COROLLARY 4.1A. Let } L_1 \text{ and } L_2 \text{ be members of } L. \text{ Then}
\end{equation*}

\begin{itemize}
\item[(i)] $L_1$ \textsuperscript{i}-degree upward dominates $L_2$
\item[(ii)] $G_k(L_1) < G_k(L_2)$ for $k = i, i+1, i+2,...$
\end{itemize}

Equation (23) implies that $P_{1k}(t) < 0$ for all $t \in (0,1]$ when $k > 0$ and thus that the $G_k$-measures satisfy the Pigou-Dalton principle of transfers for $k > 0$. Moreover, $P_{1k}(t) > 0$ for all $t \in (0,1]$ when $k > 1$. Hence all $G_k$ for $k > 1$ preserve second-degree upward Lorenz dominance. Moreover, the derivatives of $P_k$ alternate in sign up to the $(k+1)^{th}$ derivative and $P_{1k}^{(j)}(1) = 0$ for all $j \leq k$. Thus, it follows from Theorem 3.2A that $G_k$ preserves upward Lorenz dominance of degree $k$ and obeys the principles of DPTS up to and including $(k-1)^{th}$-degree. The highest degree of downside inequality averse behavior occurs as $k \to \infty$, which corresponds to the inequality averse behavior of the Rawlsian (relative) maximin criterion. Thus, $G_k$ satisfies all degrees of upward Lorenz dominance as $k \to \infty$. At the other extreme, as $k = 0$, the preference function $P_0(t) = t$, which means that $J_{h_0}$ does not obey the Pigou-Dalton principle of transfers or any principle of DPTS. The stated properties of the $G_k$-measures are summarized in the following corollary,

\begin{equation*}
\text{COROLLARY 4.2A. The extended Gini family of inequality measures defined by (10) has the following properties,}
\end{equation*}

\begin{itemize}
\item[(i)] $G_k$ preserves upward Lorenz dominance of degree $k$ and all degrees lower than $k$,
\item[(ii)] $G_k$ obeys the principle of transfers for $k > 0$,
\item[(iii)] $G_k$ obeys the Pigou-Dalton principle of transfers and the principles of DPTS up to and including $(k-1)^{th}$ degree,
\item[(iv)] If $G_k(L_1) < G_k(L_2)$ then $G_k(L_1)$ can be obtained from $G_k(L_2)$ by a sequence of downside MG$_{k-1}$-PTs',
\item[(v)] The sequence $\{G_k\}$ approaches 0 as $k \to 0$.
\end{itemize}

\textsuperscript{30} Muliere and Scarsini [33] gave an alternative proof of Corollary 4.1A
(vi) The sequence \( \{G_k\} \) approaches the Rawlsian relative maximin criterion as \( k \to \infty \).

Note that \( P_{1k} \) has negative derivatives (of any order) when \( 0 < k < 1 \). Thus, \( G_k \) for \( 0 < k < 1 \) preserves downward Lorenz dominance of all degrees.

As demonstrated by Aaberge [1] the Lorenz family of inequality measures is a subfamily of \( \{J_p : P \in P_1\} \) formed by the following family of P-functions,

\[
P_{2k}(t) = \frac{1}{k}((k+1)t - t^{k+1}), \quad k = 1, 2, \ldots
\]  

(24)

Differentiating \( P_{2k} \) defined by (24) yields

\[
P_{2k}^{(j)}(t) = \begin{cases} 
-(k+1)(k-1)(k-2)\ldots(k-j+2)t^{k-j+1}, & j = 2, 3, \ldots, k+1 \\
0 & j = k+2, k+3, \ldots
\end{cases}
\]  

(25)

By noting from (25) that the weight-function \( P_{2k} \) of the Lorenz family of inequality measures is a member of \( P^*_1 \) for \( k = i, i+1, \ldots \), we obtain the following result from Theorem 3.2B.

**COROLLARY 4.1B.** Let \( L_1 \) and \( L_2 \) be members of \( L \). Then

(i) \( L_1 \) \(^{\text{th}}\)-degree downward dominates \( L_2 \)

implies

(ii) \( D_k(L_1) < D_k(L_2) \) for \( k = i, i+1, i+2, \ldots \)

and

\[
\frac{F_{L_1}^{-1}(1)}{\mu_1} < \frac{F_{L_2}^{-1}(1)}{\mu_2}.
\]

The results of a similar evaluation of the Lorenz family of inequality measures as that carried out for the extended Gini family are summarized in the following corollary.

**COROLLARY 4.2B.** The Lorenz family of inequality measures defined by (12) has the following properties,

(i) \( D_k \) preserves downward Lorenz dominance of degree \( k \) and all degrees lower than \( k \),

(ii) \( D_k \) obeys the principle of transfers for \( k > 0 \),

(iii) \( D_k \) obeys the Pigou-Dalton principle of transfers and the principles of UPTS up to and including \((k-1)^{\text{th}}\) degree,

(iv) If \( D_k(L_1) < D_k(L_2) \) then \( D_k(L_1) \) can be obtained from \( D_k(L_2) \) by a sequence of upside \( DG_{k-1}\) PTSs'.
(v) The sequence \( \{D_k\} \) approaches 0 as \( k \to \infty \).

(vi) The sequence \( \{kD_k + 1\} \) approaches the relative minimax criterion as \( k \to \infty \).

Note that the derivatives of \( P_{2k} \) alternate in sign when \(-1 < k < 1\). Thus, \( D_k \) for \(-1 < k < 1\) preserves upward Lorenz dominance of all degrees and approaches the Rawlsian relative maximin as \( k \) approaches -1. As demonstrated by Aaberge \([1]\) \( D_k \) approaches the Bonferroni coefficient as \( k \to 0 \).

Corollary 4.1A shows that the various degrees of upward Lorenz dominance is preserved by sub-families of the extended Gini measures of inequality, which divide the integer subscript subclass of the extended Gini family into nested subfamilies. Thus, the hierarchical sequence of nested upward Lorenz dominance criteria offers a convenient computational method for identifying the largest subfamily of the integer subscript extended Gini family of inequality measures that is consistent with the actual ranking of Lorenz curves. As demonstrated by Corollary 4.1B the various degrees of downward Lorenz dominance divide the Lorenz family of inequality measures into a similar sequence of nested subfamilies.

5. SUMMARY AND DISCUSSION

This paper introduces two sequences of partial orderings for achieving complete rankings of Lorenz curves. In particular, we have examined situations where Lorenz curves intersect by introducing ranking criteria that are weaker than non-intersecting dominance (first-degree Lorenz dominance) and stronger than single measures of inequality. The proposed dominance criteria are shown to characterize nested subsets of the families of inequality measures defined by

\[
\int P'(u) dL(u) \]

where \( P' \) is the derivative of a function \( P \) that defines the inequality aversion profile of the inequality measure. The condition of first-degree Lorenz dominance corresponds to concave \( P \)-functions. By introducing higher degrees of dominance, this paper provides a method for identifying the lowest degree of dominance and the weakest restriction on the functional form of the preference function \( P \) that is needed to reach unambiguous rankings of Lorenz curves, irrespective of whether one’s social preferences is consistent with downside or upside inequality aversion. To judge the normative significance of the sequences of dominance criteria, appropriate principles of transfers and mean-“spread”-preserving transformations have been introduced. The criteria of Lorenz dominance provide convenient computational methods for ranking a set of Lorenz curves and for exploring how robust the attained ranking would be with respect to choice of rank-dependent measures of inequality. Thus, in applied work the ranking obtained by applying this approach should in general have a wider degree of support than that obtained by applying arbitrarily chosen summary measures of inequality.

To deal with the mean income income inequality trade-off, in cases where they conflict, Shorrocks \([34]\) introduced the “generalized Lorenz curve”, defined as a mean scaled-up version of the
Lorenz curve. Moreover, Shorrocks [40] obtained characterizations of social welfare functions based on first-degree dominance relations between generalized Lorenz curves. Thus, scaling up the introduced Lorenz dominance relations of this paper by the mean income (μ) and replacing the rank-dependent measures of inequality J_p defined by (2) with the rank-dependent social welfare functions \( W_p = \mu(1 - J_p) \), it can be demonstrated that the present results also apply to the generalized Lorenz curve and moreover provide convenient characterizations of the corresponding social welfare orderings.
APPENDIX

Proofs of Dominance Results

Lemma 1. Let $H$ be the family of bounded, continuous and non-negative functions on $[0,1]$ which are positive on $(0,1)$ and let $g$ be an arbitrary bounded and continuous function on $[0,1]$. Then

$$\int g(t) h(t) \, dt > 0 \quad \text{for all } h \in H$$

implies

$$g(t) \geq 0 \quad \text{for all } t \in [0,1]$$

and the inequality holds strictly for at least one $t \in (0,1)$.

The proof of Lemma 1 is known from mathematical textbooks.

The proof of the equivalence between (i) and (ii) in Theorem 2.2A is analogous to the proof for stochastic dominance in Hadar and Russel [22] but is included below for the sake of completeness.

Proof of Theorem 2.2A. Using integration by parts we have that

$$J_p(L_2) - J_p(L_1) = -P''(t) \int_0^1 (L_1(u) - L_2(u)) \, du + \int_0^1 P'''(u) \int_0^u (L_1(t) - L_2(t)) \, dt \, du.$$

Thus, if (i) holds then $J_p(L_2) > J_p(L_1)$ for all $P \in P_2$.

To prove the converse statement we restrict to preference functions $P \in P_2$ for which $P''(1) = 0$. Hence,

$$J_p(L_2) - J_p(L_1) = \int_0^1 P'''(u) \int_0^u (L_1(t) - L_2(t)) \, dt \, du$$

and the desired result it obtained by applying Lemma 1.

To prove the equivalence between (ii) and (iii) consider a case where we transfer a small amount $\gamma$ from persons with incomes $F^{-1}(s+h_1)$ and $F^{-1}(t+h_1)$ to persons with incomes $F^{-1}(s)$ and $F^{-1}(t)$, respectively, where $t$ is assumed to be larger than $s$. Then $J_p$ defined by (2) obeys the first-degree DPTS if and only if

$$P'(s) - P'(s+h_1) > P'(t) - P'(t+h_1).$$
which for small $h$ is equivalent to
\[ P'(t) - P'(s) > 0. \]

Next, inserting for $t = s + h_2$, we find, for small $h_2$, that this is equivalent to $P''(s) > 0$.

The proof of Theorem 2.2B is analogous to the proof of Theorem 2.2A and is based on the expression
\[ J_p(L_2) - J_p(L_1) = -P'(0) \int_0^1 (L_1(t) - L_2(t)) \, dt - \int_0^1 P''(u) \int_u^1 (L_1(t) - L_2(t)) \, dt \, du \]
which is obtained by using integration by parts. Thus, by arguments like those in the proof of Theorem 2.2A the results of Theorem 2.2B are obtained.

**Proof of Proposition 2.1A.** The statement (i) implies (ii) follows from Theorem 2.2A.

To prove the converse statement assume that (ii) holds and that $L_1$ and $L_2$ cross at $u = a$. Then the following inequalities hold,
\[ \int_a^0 (L_1(u) - L_2(u)) \, du > 0 \quad \text{(A1)} \]
and
\[ \int_0^1 (L_1(u) - L_2(u)) \, du = \frac{1}{2} (G_2 - G_1) \geq 0. \quad \text{(A2)} \]
Since $L_1$ and $L_2$ cross only once (A1) and (A2) imply that
\[ \int_0^u (L_1(u) - L_2(u)) \, du \geq 0 \text{ for all } u \in [0,1] \]
and the inequality holds strictly for some $u$, and the desired result is obtained by applying Theorem 2.2A.

**Proof of Theorem 2.3.** Assume that
\[ (i) \quad \int_0^u L_1(t) \, dt \geq \int_0^u L_2(t) \, dt \text{ for all } u \in [0,1] \]
and the inequality holds strictly for some $u \in (0,1)$ and that $L_1$ and $L_2$ intersect $m$ times, where $m$ is an arbitrary integer. Since $\frac{1}{0} \int L_1(t) \, dt = \frac{1}{0} \int L_2(t) \, dt$ and $L_1$ and $L_2$ intersect $m$ times, then (i) implies that $L_1(u) \geq L_2(u)$ for $u \in [0,a_1]$ and $L_1(u) \leq L_2(u)$ for $u \in [a_m,1]$ where $a_i$ is the first intersection point and $a_m$ is the last intersection point. Accordingly, $m$ has to be odd. For convenience we let $m = 2n - 1$, where $n$ is an arbitrary integer, and let $a_j$, $j = 1,2,...,2n-1$ denote the $2n-1$ $u$-values where $L_1$ and $L_2$ intersect, i.e., $L_1(a_j) = L_2(a_j)$ for $j = 1,2,...,2n-1$. Thus, we have that

$$L_1(u) \begin{cases} \geq L_2(u) \quad \text{for} \quad u \in [a_{2j-2},a_{2j-1}] \\ \leq L_2(u) \quad \text{for} \quad u \in [a_{2j-1},a_{2j}] \end{cases} \quad (A3)$$

for $j = 1,2,...,n$ where $0 < a_1 < a_2 < ... < a_{2n-1} < 1$, $a_0 = 0$ and $a_{2n} = 1$. Furthermore, let $L_{1j}$ and $L_{2j}$ be the Lorenz curves defined by

$$L_{1j}(u) = \begin{cases} L_1(u) \quad \text{for} \quad u \in [0,a_{2j-2}] \quad \text{and} \quad u \in [a_{2j-1},1] \\ L_2(u) \quad \text{for} \quad u \in [a_{2j-2},a_{2j-1}] \end{cases} \quad (A4)$$

and

$$L_{2j}(u) = \begin{cases} L_1(u) \quad \text{for} \quad u \in [0,a_{2j-1}] \quad \text{and} \quad u \in [a_{2j},1] \\ L_2(u) \quad \text{for} \quad u \in [a_{2j-1},a_{2j}] \end{cases} \quad (A5)$$

for $j = 1,2,...,n$.

Then it follows from (A3) that

$$L_{1j}(u) \leq L_1(u) \quad \text{for all} \quad u \in [0,1]$$

and

$$L_{2j}(u) \geq L_1(u) \quad \text{for all} \quad u \in [0,1]$$

for $j = 1,2,...,n$.

By applying Theorem 2.1 we get that

$$L_{1j}(u) \leq L_1(u) \quad \text{for all} \quad u \in [0,1]$$

if and only if $L_1$ can be obtained from $L_{1j}$ by a sequence of Pigou-Dalton progressive transfers, and
$L_{2j}(u) \geq L_j(u)$ for all $u \in [0,1]$ 

if and only if $L_i(u)$ can be obtained from $L_{2j}$ by a sequence of Pigou-Dalton regressive transfers.

Next, by noting that

$$L_i(u) - L_{2j}(u) = \sum_{j=1}^{2} \sum_{j=1}^{n} \left( L_i(u) - L_0(u) \right)$$

we then have that $L_1$ can be obtained from $L_2$ by sequences of Pigou-Dalton progressive and regressive transfers.

It follows from (A3) that the $2n$ segments formed by the $2n-1$ intersections may be arranged in $n$ pairs where $L_i(u)$ dominates $L_{2j}(u)$ when $u \in [a_{2j-1}, a_{2j+1}]$ and $L_{2j}(u)$ dominates $L_i(u)$ when $u \in [a_{2j-1}, a_{2j}]$, $j = 1, 2, ..., n$. Thus, since $a_{2j-2} < a_{2j-1} < a_{2j}$ it follows for each pair of segments that the progressive transfers occur for lower income levels than the regressive transfers. Moreover, since

$$\int_0^1 \sum_{j=1}^{n} \sum_{j=1}^{2} \left( L_i(u) - L_0(u) \right) du =$$

$$= \sum_{j=1}^{n} \int_{a_{2j-2}}^{a_{2j+1}} \left( L_i(u) - L_{2j}(u) \right) du + \int_{a_{2j-2}}^{a_{2j+1}} \left( L_i(u) - L_{2j}(u) \right) du$$

$$= \int_0^1 \left( L_i(u) - L_{2j}(u) \right) du = 0$$

we get that

$$\sum_{j=1}^{n} \int_{a_{2j-2}}^{a_{2j+1}} \left( L_i(u) - L_{2j}(u) \right) du = \sum_{j=1}^{n} \int_{a_{2j-2}}^{a_{2j+1}} \left( L_{2j}(u) - L_i(u) \right) du .$$

(A7)

Thus, in order to fulfill the condition of equal Gini coefficients the sequence of Pigou-Dalton progressive transfers captured by the left side of equation (A7) has to be matched by a corresponding sequence of Pigou-Dalton regressive transfers captured by the right side of equation (A7). Accordingly, we have found that $L_1$ can be obtained from $L_2$ by a downside mean-Gini-preserving transformation.

To prove that (iii) implies (i) we will rely on Theorem 2.2A and follow the line of reasoning used by Zoli [49] for the proof of the “only if part” of his Proposition 3.

Consider the family $J_P$ defined by (2) for $P \in \mathbb{P}_1$, i.e. for increasing concave $P$, let $F_1$ and $F_2$ be discrete distributions with Lorenz curves $L_1$ and $L_2$, and assume that $L_1$ can be obtained from $L_2$ by a downside mean-Gini-preserving transformation where $N = 1$. Thus we consider a Pigou-Dalton
progressive transfer $\delta$ from a person with income $F_2^{-1}(s + h)$ to a person with income $F_2^{-1}(s)$ and a Pigou-Dalton regressive transfer $\gamma$ from a person with income $F_2^{-1}(t)$ to a person with income $F_2^{-1}(t + \tilde{h})$, where $s < t$. Since these transfers are assumed to leave the Gini coefficient unchanged it follows from (3) that

$$\delta[(1-s)-(1-s-h)] = \gamma[(1-t)-(1-t-\tilde{h})]$$

which is equivalent to

$$\delta h = \gamma \tilde{h}. \quad (A8)$$

Furthermore, from (2) we have that a downside MGPT reduces inequality, i.e. $J_p(L_1) < J_p(L_2)$, if and only if

$$\delta[P'(s) - P'(s + h)] > \gamma[P'(t) - P'(t + \tilde{h})]. \quad (A9)$$

Inserting for (A8) in (A9) yields

$$\frac{P'(s) - P'(s + h)}{h} > \frac{P'(t) - P'(t + \tilde{h})}{\tilde{h}}$$

which for small $h$ and $\tilde{h}$ is equivalent to

$$P'(t) - P'(s) > 0.$$

Next, inserting for $t = s + h_1$, we find, for small $h_1$, that this inequality is equivalent to $P''(s) > 0$. Then it follows from Theorem 2.2A that $L_1$ second-degree upward dominates $L_2$.

The equivalence between (i) and (ii) follows by noting that

$$\int_u^1 [(1-L_1(t)) - (1-L_2(t))]dt = \int_u^1 (L_2(t) - L_1(t))dt = $$

$$\int_0^u (L_2(t) - L_1(t))dt + \int_u^0 (L_1(t) - L_2(t))dt = \int_0^u (L_1(t) - L_2(t))dt$$

when $L_1$ and $L_2$ have equal Gini coefficients.

To prove that (ii) implies (iv) we follow the line of reasoning used for proving that (i) implies (iii). To this end it is convenient to introduce the Lorenz curves $\tilde{L}_{i1}$ and $\tilde{L}_{i2}$ defined by
\[ \tilde{L}_n(u) = \begin{cases} L_2(u) & \text{for } u \in [0, a_{2i-2}] \text{ and } u \in [a_{2i-1}, 1] \\ L_1(u) & \text{for } u \in [a_{2i-2}, a_{2i-1}] \end{cases} \] (A10)

and

\[ \tilde{L}_{2i}(u) = \begin{cases} L_2(u) & \text{for } u \in [0, a_{2i-1}] \text{ and } u \in [a_{2i}, 1] \\ L_1(u) & \text{for } u \in [a_{2i-1}, a_{2i}] \end{cases} \] (A11)

for \( i = 1, 2, ..., n \).

Then it follows from (A3) that

\[ \tilde{L}_{ij}(u) \geq L_2(u) \text{ for all } u \in [0,1] \]

and

\[ \tilde{L}_{2j}(u) \leq L_2(u) \text{ for all } u \in [0,1] \]

for \( j = 1, 2, ..., n \).

By applying Theorem 2.1 we thus get that \( L_2 \) can be obtained from \( \tilde{L}_{ij} \) by a sequence of Pigou-Dalton regressive transfers and from \( \tilde{L}_{2j} \) by a sequence of Pigou-Dalton progressive transfers. Moreover, by observing that

\[ \left( L_2 - L_1 \right) = \sum_{i=1}^{2n} \sum_{j=1}^{n} \left( L_2(u) - \tilde{L}_{ij}(u) \right) \]

we thus have that \( L_2 \) can be obtained from \( L_1 \) by sequences of Pigou-Dalton regressive and progressive transfers. As noted above \( L_2(u) \) dominates \( L_1(u) \) when \( u \in [a_{2j-1}, a_{2j}] \) and \( L_2(u) \) is dominated by \( L_1(u) \) when \( u \in [a_{2j-2}, a_{2j-1}] \), \( j = 1, 2, ..., n \). Thus, by arranging the adjacent segments \([a_{2j-2}, a_{2j-1}]\) and \([a_{2j-1}, a_{2j}]\) in \( n \) pairs we find for each pair of segments that the regressive transfers occur for lower income levels than the progressive transfers. Moreover, since

\[ \int_0^1 \sum_{i=1}^{2n} \sum_{j=1}^{n} \left( L_2(u) - \tilde{L}_{ij}(u) \right) du = \sum_{i=1}^{2n} \left[ \int_{a_{2i-2}}^{a_{2i-1}} \left( L_2(u) - L_1(u) \right) du + \int_{a_{2i-1}}^{a_{2i}} \left( L_2(u) - L_1(u) \right) du \right] = \]

\[ \int_0^1 \left( L_2(u) - L_1(u) \right) du = 0 \]

we get that
\[ \sum_{j=1}^{n} \int_{a_{2j-1}}^{a_{2j}} (L_1(u) - L_2(u)) \, du = \sum_{j=1}^{n} \int_{a_{2j-1}}^{a_{2j}} (L_2(u) - L_1(u)) \, du . \quad (A13) \]

Thus, in order to fulfill the condition of equal Gini coefficients the sequence of Pigou-Dalton regressive transfers captured by the left side of equation (A13) has to be matched by a corresponding sequence of Pigou-Dalton progressive transfers captured by the right side of equation (A13).

To prove that (iv) implies (ii) we consider a Pigou-Dalton regressive transfer \( \delta \) from a person with income \( F_1^{-1}(s) \) to a person with income \( F_1^{-1}(s+h) \) and a Pigou-Dalton progressive transfer \( \gamma \) from a person with income \( F_1^{-1}(t+h) \) to a person with income \( F_1^{-1}(t) \) where \( s < t \). Since these transfers are assumed to leave the Gini coefficient unchanged it follows that the condition (A8) has to be fulfilled. Furthermore, it follows from (2) that an upside MGPT reduces inequality, i.e. \( J_p(L_2) < J_p(L_1) \), if and only if

\[ \gamma \left[ P'(t) - P'(t+h) \right] > \delta \left[ P'(s) - P'(s+h) \right] \]

which by inserting for the condition (A8) is equivalent to

\[ \frac{P'(t) - P'(t+h)}{h} > \frac{P'(s) - P'(s+h)}{h} \]

which for small \( \tilde{h} \) and \( h \) is equivalent to

\[ P'(t) - P'(s) < 0 . \]

Next, inserting for \( t = s + h_1 \), we find, for small \( h_1 \), that this inequality is equivalent to \( P''(s) < 0 \). Then it follows from Theorem 2.2B that \( L_2 \) second-degree downward dominates \( L_1 \).

**Proof of Theorem 3.1A.** To examine the case of \( i^{th} \)-degree upward Lorenz dominance we integrate \( J_p(L_2) - J_p(L_1) \) by parts \( i \) times,

\[ J_p(L_2) - J_p(L_1) = \sum_{i=1}^{i} (-1)^{i-1} \int_{0}^{1} P^{(i)}(u) \left( L_1^{(i)}(u) - L_2^{(i)}(u) \right) \, du + (-1)^{i} \int_{0}^{1} P^{(i+1)}(u) \left( L_1^{(i)}(u) - L_2^{(i+1)}(u) \right) \, du \quad (A14) \]

and use this expression in constructing the proof of the equivalence between (i) and (ii).

Assume first that (i) in Theorem 3.1A is true, i.e.

\[ L_1^{(i)}(u) - L_2^{(i)}(u) \geq 0 \text{ for all } u \in [0,1] \]
and > holds for at least one \( u \in (0,1) \).

Then \( J_p(L_2) > J_p(L_1) \) for all \( P \in P_{ii}^* \).

Conversely, assume that

\[
J_p(L_2) > J_p(L_1) \quad \text{for all} \quad P \in P_{ii}^* .
\]

For this family of preference functions we have that

\[
J_p(L_2) - J_p(L_1) = (-1)^i \int_0^1 p^{(i+1)}(u) \left( L_i^1(u) - L_i^2(u) \right) dt .
\]

Then, as demonstrated by Lemma 1, the desired result can be obtained by a suitable choice of \( P \in P_{ii}^* \).

**Proof of Theorem 3.2A.** Assume that (i) in Theorem 3.2A is true, i.e.

\[
L_i(u) - L_i^1(u) \geq 0 \quad \text{for all} \quad u \in [0,1]
\]

and > holds for at least one \( u \in (0,1) \). Then it follows from Theorem 3.1A that \( J_p(L_2) > J_p(L_1) \) for all \( P \in P_{ii}^* \) such that \((1)^i p^{(i+1)}(t) > 0 \) for \( t \in (0,1) \), \( j=1,2,...,i \) since this family of P-functions is a subfamily of \( P_{ii}^* \).

Conversely, assume that \( J_p(L_2) > J_p(L_1) \) for all \( P \in P_{ii}^* \) such that \((1)^i p^{(i+1)}(t) > 0 \) for \( t \in (0,1) \). For this family of P-functions we have that

\[
J_p(L_2) - J_p(L_1) = (-1)^i \int_0^1 p^{(i+1)}(u) \left( L_i(u) - L_i^1(u) \right) du .
\]

and the desired result is obtained by applying Lemma 1.

A proof by mathematical induction will be used to prove the equivalence between (ii) and (iii). To this end it is convenient to introduce the following notation. Let \( H_1 \), \( H_2 \) and \( H_{j}\) be defined by

\[
H_1(v, h) = P'(v) - P'(v + h) , \quad (A15)
\]

\[
H_2(s, t, h) = H_1(s, h) - H_1(t, h) \quad (A16)
\]

and

\[
H_{j}(s,t,h_1,...,h_j) = H_j(s,t,h_1,...,h_{j-1}) - H_j(s+h_j,t+h_j,h_1,...,h_{j-1}) , \quad j = 2,3,... . \quad (A17)
\]

Moreover, let
\[ H_2^{(j)}(s, t) = \lim_{h_j \to 0} \frac{1}{h_j} H_2(s, t, h_j) \]  

(A18)

and

\[ H_{j+1}^{(j)}(s, t) = \lim_{h_j \to 0} \lim_{h_{j+1} \to 0} \frac{1}{h_j h_{j+1}} H_{j+1}(s, t, h_1, h_2, \ldots, h_j). \]  

(A19)

It follows from Theorems 2.1 and 2.2A that \( J_P \) obeys the Pigou-Dalton principle of transfers and the first-degree DPTS iff \( P'(t) < 0 \) and \( P''(t) > 0 \). From (14), definition (2) of \( J_P \) and (A15)-(A19) we then get that \( J_P \) obeys the second-degree DPTS iff

\[ H_3^{(2)}(s, t) > 0 \quad \text{for} \quad s < t. \]  

(A20)

Inserting for (A17), (A16) and (A15) in (A19) for \( j = 2 \) yields

\[ H_3^{(2)}(s, t) = \lim_{h_2 \to 0} \lim_{h_1 \to 0} \frac{1}{h_1 h_2} H_3(s, t, h_1, h_2) = \]

\[ \lim_{h_2 \to 0} \lim_{h_1 \to 0} \frac{1}{h_1 h_2} \left( H_2(s, t, h_1) - H_2(s + h_2, t + h_2, h_1) \right) = \]

\[ \lim_{h_2 \to 0} \left( H_2^{(1)}(s, t) - H_2^{(1)}(s + h_2, t + h_2) \right) = \]

\[ \lim_{h_1 \to 0} \frac{1}{h_1} \left( \left[ P'(s - P'(s + h_1) - (P'(t) - P'(t + h_1)) - \right. \right. \]

\[ \left. \left. \left. \left[ P'(s + h_2) - P'(s + h_1 + h_2) - (P'(t + h_2) - P'(t + h_1 + h_2)) \right] \right] \right] = \lim_{h_2 \to 0} \left[ -P'(s + h_2) - P'(s + h_2) - (P'(t + h_2) - P'(t + h_2)) \right] = P^{(3)}(s) - P^{(3)}(t). \]

Inserting for \( t = s + h \), we find, for small \( h \), that this is equivalent to \( P^{(4)}(s) < 0 \).

Next, assume that

\[ H_j^{(j)}(s, t) = (-1)^{j-1} \left( P^{(j)}(s) - P^{(j)}(t) \right). \]  

(A21)

It follows from Theorem 2.2A and the proof above that (A21) is true for \( j \) equal to 2 and 3.
\[
H_{j_1}(s,t) = \lim_{h_j \to 0} \lim_{h_{j-1} \to 0} \frac{1}{\prod_{k=1}^{j-1} h_k} \left( H_j(s,t,h_1,h_2,...,h_{j-1}) - H_j(s + h_j, t + h_j, h_1, h_2, ..., h_{j-1}) \right) = \\
\lim_{h_j \to 0} \lim_{h_{j-1} \to 0} \frac{1}{\prod_{k=2}^{j-1} h_k} \left( H_j(s,t,h_1,h_2,...,h_{j-1}) - H_j(s + h_j, t + h_j, h_2, ..., h_{j-1}) \right) = \\
\lim_{h_j \to 0} \frac{1}{h_j} \left( H_j^{(j-1)}(s,t) - H_j^{(j-1)}(s + h_j, t + h_j) \right),
\]

which by inserting for (A21) yields

\[
H_{j_1}(s,t) = (-1)^j \left( P^{(i_1)}(s) - P^{(j)}(t) \right).
\]

Thus, (A21) is proved to be true by induction.

Since \( J_p \) defined by (2) obeys the \((i-1)\)th-degree DPTS if and only if

\[
H_{i}^{(i-1)}(s,t) > 0 \text{ for } s < t
\]

we get from (A21) that this condition is equivalent to

\[
(-1)^j P^{(i_1)}(s) > 0.
\]

**Proof of Theorem 3.3A.** By inserting for (9) in (A14) we get that

\[
J_p(L_2) - J_p(L_1) = \sum_{j=2}^{i} (-1)^{j-1} P^{(j)}(L_j) (G_{j-1}(L_j) - G_{j-1}(L_1)) + (-1)^{\frac{i}{0}} \int_{0}^{1} P^{(i)}(u) \left(L^{i}(u) - L^{i}(u)\right) \, du \quad \text{(A22)}
\]

Assume first that (i) of Theorem 3.3A is true. Then \( J_p(L_2) > J_p(L_1) \) for all \( P \in P_{ii}^{***} \).

Conversely, assume that

\[
J_p(L_2) > J_p(L_1) \text{ for all } P \in P_{ii}^{***}.
\]

Then this statement holds for the subfamily of \( P_{ii}^{***} \) for which \( P^{(j)}(1) = 0 \) for \( j = 2,3,\ldots,i \). For this particular family of preference functions we get that

\[
J_p(L_2) - J_p(L_1) = (-1)^{\frac{i}{0}} \int_{0}^{1} P^{(i)}(u) \left(L^{i}(u) - L^{i}(u)\right) \, du.
\]

By applying Lemma 1 we get that \( L_i \) \(i\th\)-degree upward dominates \( L_2 \).

Next, consider the subfamily of preference functions defined by
\[ P_k(t) = 1 - (1 - t)^{k+1}, \quad k = 1, 2, \ldots, i - 1. \] (A23)

By observing that \( P_k \in P_{ii}^{++} \) we find by inserting for (A23) in \( J_P \) that

\[ 0 < J_{R_k}(L_2) - J_{R_k}(L_1) = G_k(L_2) - G_k(L_1) \]

for \( k = 1, 2, \ldots, i - 1 \).

**The proofs of Theorems 3.1B, 3.2B and 3.3B** can be constructed by following exactly the line of reasoning used in the proofs of Theorems 3.1A and 3.3A. The proofs use the following expression,

\[ J_P(L_2) - J_P(L_1) = -\sum_{j=2}^{i} P^{(j)}(0) \left( \tilde{L}_2(0) - \tilde{L}_1(0) \right) - \int_0^1 P^{(j+1)}(u) \left( \tilde{L}_2(u) - \tilde{L}_1(u) \right) du, \] (A24)

which is obtained by using integration by parts \( i \) times.

**Proof of Theorem 3.4A.**

To prove the equivalence between (i) and (ii) we draw on the proof of Theorem 3.3A. Under the condition of equal Gini coefficients and \( G_j(L_1) = G_j(L_2), \quad j = 2, 3, \ldots, i \) we get from (A22) that

\[ J_P(L_2) - J_P(L_1) = (-1)^{i+1} \int_0^1 P^{(j+2)}(u) \left( \tilde{L}_2^{(i+1)}(u) - \tilde{L}_1^{(i+1)}(u) \right) du, \] (A25)

and the desired result is obtained by applying Lemma 1.

Next, we will prove that (i) implies (iii). Assume that

\( (i) \quad \int_0^u L_1^{(i)}(t) dt \geq \int_0^u L_2^{(i)}(t) dt \) for all \( u \in [0,1] \)

and the inequality holds strictly for some \( u \in (0,1) \) and \( L_1^{(i)} \) and \( L_2^{(i)} \) intersect \( m \) times, where \( m \) is an arbitrary integer. Since \( G_i(L_1) = G_i(L_2) \) is equivalent to \( \int_0^1 L_1^{(i)}(t) dt = \int_0^1 L_2^{(i)}(t) dt \) and \( L_1^{(i)} \) and \( L_2^{(i)} \) intersect \( m \) times, then \( L_1^{(i)}(u) \geq L_2^{(i)}(u) \) for \( u \in [0,b_1] \) and \( L_1^{(i)}(u) \leq L_2^{(i)}(u) \) for \( u \in [b_m,1] \) where \( b_i \) is the first intersection point and \( b_m \) is the last intersection point. Accordingly \( m \) have to be odd. For convenience we let \( m = 2n - 1 \), where \( n \) is an arbitrary integer, and let \( b_i, i = 1, 2, \ldots, 2n - 1 \) denote the \( 2n - 1 \) \( u \)-values where \( L_1^{(i)} \) and \( L_2^{(i)} \) intersect, i.e. \( L_1^{(i)}(b_j) = L_2^{(i)}(b_j) \) for \( j = 1, 2, \ldots, 2n - 1 \). Thus, we have that
\[ L_i^j(u) = \begin{cases} 
 L_i^j(u) & \text{for } u \in \left[0, b_{2j-2}\right] \text{ and } u \in \left[b_{2j-1}, 1\right] \\
 L_2^j(u) & \text{for } u \in \left[b_{2j-2}, b_{2j-1}\right] 
\end{cases} \] (A26)

for \( j = 1, 2, \ldots, n \), where \( 0 < b_1 < b_2 < \ldots < b_{2n-1} < 1 \), \( b_0 = 0 \) and \( b_{2n} = 1 \). Furthermore, let \( L_{ij} \) and \( L_{2j} \) be the \( L^i \)-curves defined by

\[ L_{ij}^j(u) = \begin{cases} 
 L_i^j(u) & \text{for } u \in \left[0, b_{2j-2}\right] \text{ and } u \in \left[b_{2j-1}, 1\right] \\
 L_2^j(u) & \text{for } u \in \left[b_{2j-2}, b_{2j-1}\right] 
\end{cases} \] (A27)

and

\[ L_{2j}^j(u) = \begin{cases} 
 L_i^j(u) & \text{for } u \in \left[0, b_{2j-1}\right] \text{ and } u \in \left[b_{2j}, 1\right] \\
 L_2^j(u) & \text{for } u \in \left[b_{2j-1}, b_{2j}\right] 
\end{cases} \] (A28)

for \( j = 1, 2, \ldots, n \).

Then it follows from (A26) that

\[ L_{ij}^j(u) \leq L_{ij}^j(u) \text{ for all } u \in [0, 1] \]

and

\[ L_{2j}^j(u) \geq L_{ij}^j(u) \text{ for all } u \in [0, 1] \]

for \( j = 1, 2, \ldots, n \).

Let us first consider the case where \( i = 2 \), i.e. third degree upward Lorenz dominance. By applying Theorem 2.3 we have that

\[ L_{ij}^j(u) \leq L_{ij}^j(u) \text{ for all } u \in [0, 1] \]

if and only if \( L_1 \) can be obtained from \( L_{ij} \) by a downside mean-Gini-preserving transformation (downside MGPT), and

\[ L_{ij}^j(u) \geq L_{ij}^j(u) \text{ for all } u \in [0, 1] \]

if and only if \( L_i(u) \) can be obtained from \( L_{ij} \) by an upside MGPT.

Next, by noting that

\[ L_2^j(u) - L_2^j(u) = \sum_{k=1}^{2} \sum_{j=1}^{n} \left(L_i^j(u) - L_{ij}^j(u)\right) \] (A29)
we then have that \( L_1 \) can be obtained from \( L_2 \) by a combination of a downside and an upside MGPT. It follows from (A26) for \( i = 2 \) that the \( 2n \) segments formed by the \( 2n - 1 \) intersections can be arranged in \( n \) pairs where \( L_i(u) \) second-degree upward dominates \( L_2(u) \) when \( u \in [b_{2j-2}, b_{2j-1}] \) and \( L_i(u) \) second-degree downward dominates \( L_2(u) \) when \( u \in [b_{2j-2}, b_{2j}] \). Thus, since \( b_{2j-2} < b_{2j-1} < b_{2j} \) it follows for each pair of segments that the downside MGPT occurs for lower income levels than the upside MGPT. Moreover, since

\[
\int_0^1 \sum_{k=1}^n \sum_{j=1}^{a} \left( L_1^*(u) - L_2^*(u) \right) du = \sum_{j=1}^{a} \int_{b_{2j-2}}^{b_{2j+1}} \left( L_1^*(u) - L_2^*(u) \right) du + \int_{b_{2j+1}}^{b_{2j}} \left( L_1^*(u) - L_2^*(u) \right) du = \int_0^1 \left( L_1^*(u) - L_2^*(u) \right) du = G_2(L_2) - G_2(L_1) = 0
\]

we get that

\[
\sum_{j=1}^{a} \int_{b_{2j-2}}^{b_{2j+1}} \left( L_1^*(u) - L_2^*(u) \right) du = \sum_{j=1}^{a} \int_{b_{2j+1}}^{b_{2j}} \left( L_1^*(u) - L_2^*(u) \right) du. \tag{A30}
\]

Thus, in order to fulfill the condition of equal \( G_2 \)-coefficients the sequence of downside MGPTs' captured by the left side of equation (A30) has to be matched by a corresponding sequence of upside MGPTs' captures by the right side of equation (A30). Accordingly, we have found that \( L_1 \) can be obtained from \( L_2 \) by a downside MGPT.

By following the line of reasoning used to prove that third-degree upward Lorenz dominance implies that the dominating Lorenz curve can be obtained from the dominated Lorenz curve by a downside MGPT we assume that a Lorenz curve \( L_1 \) that \( i^{th} \)-degree upward dominates a Lorenz curve \( L_2 \) can be obtained from \( L_2 \) by a combination of a downside and an upside MGPT. Then it follows from (A26) that the \( 2n \) segments formed by the \( 2n - 1 \) intersections of \( L_i^* \) and \( L_2^* \) can be arranged in \( n \) pairs where \( L_i(u) \) \( i^{th} \)-degree upward dominates \( L_2(u) \) when \( u \in [b_{2j-2}, b_{2j-1}] \) and \( L_i(u) \) \( i^{th} \)-degree downward dominates \( L_2(u) \) when \( u \in [b_{2j-1}, b_{2j}] \). Thus, since \( b_{2j-2} < b_{2j-1} < b_{2j} \) it follows for each pair of segments that the downside MGPT occurs for lower income levels than the upside MGPT. Moreover, since
\[
\int_0^1 \sum_{k=1}^n \sum_{j=1}^n (L_i^k(u) - L_i^j(u)) \, du = \\
\sum_{j=1}^n \left[ \int_{b_{j+1}}^{b_j} (L_i^1(u) - L_i^j(u)) \, du + \int_{b_{j+1}}^{b_j} (L_i^j(u) - L_i^{j+1}(u)) \, du \right] = \\
\int_0^1 (L_i^1(u) - L_i^j(u)) \, du = G_i(L_2) - G_i(L_1) = 0
\]

we get that

\[
\sum_{j=1}^n \int_{b_{j+1}}^{b_j} (L_i^1(u) - L_i^j(u)) \, du = \sum_{j=1}^n \int_{b_{j+1}}^{b_j} (L_i^j(u) - L_i^{j+1}(u)) \, du .
\]

(A31)

Thus, in order to fulfill the condition of equal \(G_i\)-coefficients the sequence of downside MG\(_{1 \to i}\)PTs’ captured by the left side of equation (A31) has to be matched by a corresponding sequence of upside MG\(_{1 \to i}\)PTs’ captured by the right side of equation (A31). Accordingly, we have found that \(L_1\) can be obtained from \(L_2\) by a downside MG\(i\)PT.

A proof by mathematical induction will be used to prove that (iii) implies (i). To this end it is convenient to introduce the following notation.

Let

\[
A_j(i,k) = \delta_j \left( (1-s_i)^k - (1-s_j-h) \right) - \gamma_j \left( (1-t_i)^k - (1-t_j-h) \right),
\]

(A32)

\[
A_j(i,k) = A_{j-1}(i,k+1) - A_{j-1}(i+a(i),k+1),
\]

(A33)

\[
T_{p}(2,i) = \delta_j \left( P'(s_i) - P'(s_i+h) \right) - \gamma_j \left( P'(t_i) - P'(t_i+h) \right)
\]

(A34)

and

\[
T_{p}(i+1,j) = T_{p}(i,j) - T_{p}(i+a(i+1))
\]

(A35)

where

\[
a(i) = 2^{i-3}, \ i = 3, 4, \ldots
\]

(A36)

Consider the family \(J_p\) defined by (2) where \(F_1\) and \(F_2\) are discrete distributions with Lorenz curves \(L_1\) and \(L_2\), and assume that \(L_1\) can be obtained from \(L_2\) by a sequence of downside MG\(i\)PTs’.

Thus, we consider pairs \((\delta_j, \gamma_j)\) of Pigou-Dalton progressive/regressive transfers \(\delta_j\) from a person with
income $F_{z}^{-1}(s_{j} + h_{j})$ to a person with income $F_{z}^{-1}(s_{j})$ and $\gamma_{j}$ from a person with income $F_{z}^{-1}(t_{j})$ to a person with income $F_{z}^{-1}(t_{j} + \tilde{h}_{j})$, where $s_{j} < t_{j}$, $s_{j} < s_{j+1}$ and $t_{j} < t_{j+1}$. Since the transfers are assumed to leave the Gini coefficient unchanged it follows from (3) that

$$A_{2}(j,1) = 0,$$

which from (A32) is found to be equivalent to

$$\delta_{j}h_{j} = \gamma_{j}\tilde{h}_{j}.$$  \hspace{1cm} (A37)

The condition of fixed $G_{2}$-coefficients is equivalent to the following requirement

$$A_{j}(j,1) = A_{2}(j,2) - A_{2}(j+1,2) = 0,$$

which by inserting for (A37) is equivalent to

$$\delta_{j}h_{j} = \gamma_{j}\tilde{h}_{j}.$$  \hspace{1cm} (A38)

By proceeding for $i = 4$ we get that the condition of fixed $G_{3}$-coefficients is equal to the requirement

$$A_{4}(j,1) = A_{3}(j,2) - A_{3}(j+1,2) = 0,$$

which by inserting for (A37) and (A38) (for small $h_{j}, \tilde{h}_{j}, h_{j+1}, \tilde{h}_{j+1}, h_{j+2}, \tilde{h}_{j+2}, h_{j+3}$ and $\tilde{h}_{j+3}$) is equivalent to

$$\delta_{j}h_{j}(t_{j} - s_{j}) = \delta_{j+1,h_{j+1}}(t_{j+1} - s_{j+1}).$$  \hspace{1cm} (A38)

which for small $t_{j+1} - s_{j+1}$ for $t = 0,1,2,3$ is found to be equivalent to
\[
\delta_j h_j(t_j - s_j)(s_{j+1} - s_j) = \delta_{j+1} h_{j+2}(t_{j+2} - s_{j+2})(s_{j+3} - s_{j+2}).
\]

Similarly, the condition of unchanged \(G_i\)-coefficients is captured by the requirement

\[
A_{j+1}(j,1) = A_j(j,2) - A_i(j + a(i + 1),2) = 0,
\]

which by inserting for the conditions of equal \(G_j\)-coefficients, \(j=1,2,...,i-1\) for small \(s_{j+1} - s_{j+1}\) is found to be equivalent to

\[
\delta_j h_j(t_j - s_j) \prod_{i=1}^{j+2} (s_{j+1} - s_j) = \delta_{j+1} h_{j+1}(t_{j+1} - s_{j+1})(s_{j+2} - s_{j+1}) \prod_{i=1}^{j+2} (s_{j+1} - s_{j+1}). 
\] (A39)

The case for \(i=1\) (second-degree Lorenz dominance) was proved for Theorem 2.3. Thus, let us firstly consider the case \(i=2\). Inserting for (A37) in \(T_p(3,1)\) defined by (A35) we find that a downside \(MG_{2PT}\) reduces inequality, i.e. \(J_p(L_1) < J_p(L_2)\), if and only if

\[
T_p(3,1) = \delta_j h_j(-P^*(s_j) + P^*(t_j)) - \delta_{j+1} h_{j+1}(-P^*(s_{j+1}) - P^*(t_{j+1})) > 0,
\]

which by inserting for (A38) is equivalent to

\[
\frac{P^*(t_j) - P^*(s_j)}{t_j - s_j} > \frac{P^*(t_{j+1}) - P^*(s_{j+1})}{t_{j+1} - s_{j+1}},
\]

which for small \(t_j - s_j\) and \(t_{j+1} - s_{j+1}\) is equivalent to

\[
P^*(s_{j+1}) - P^*(s_j) < 0,
\]

which for small \(s_{j+1} - s_j\) is equivalent to

\[
P^{(i)}(s_j) < 0.
\]

Next, we have that a downside \(MG_{iPT}\) reduces inequality if and only if

\[
T_p(i+1,1) = T_p(i,1) - T_p(i,a(i+1)+1) > 0,
\]

which by inserting for (A37) in \(T_p(2,1)\), (A38) in \(T_p(3,1)\) and so forth is found to be equivalent to
\[
\delta h_i(t_i - s_i) \prod_{i=1}^{i-1} (s_{i+1} - s_i) \left[ -1 \left( P^{(i)}(s_{i+1}) - P^{(i)}(s_i) \right) \right] > 0,
\]

which by inserting for (A39) is equivalent to

\[
\delta h_i(t_i - s_i) \prod_{i=1}^{i-1} (s_{i+1} - s_i) \left[ -1 \left( \frac{P^{(i)}(s_{i+1}) - P^{(i)}(s_i)}{s_{i+1} - s_i} - \frac{P^{(i)}(s_{d(i+1)+1}) - P^{(i)}(s_{d(i+1)+1})}{s_{d(i+1)+1} - s_{d(i+1)+1}} \right) \right] > 0,
\]

which for small \( s_{i+2} - s_i \) and \( s_{d(i+1)+1} - s_{d(i+1)+1} \) is equivalent to

\[
\delta h_i(t_i - s_i) \prod_{i=1}^{i-2} (s_{i+1} - s_i) \left[ -1 \left( P^{(i)}(s_i) - P^{(i)}(s_{d(i+1)+1}) \right) \right] > 0,
\]

which for small \( s_{d(i+1)+1} - s_i \) is equivalent to

\[
(-1)^{i+1} P^{(i+2)}(s_i) > 0.
\]

REFERENCES


