Why Rent When You Can Buy?  
A Theory of Repurchase Agreements*

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November 22, 2011

PRELIMINARY AND INCOMPLETE

Abstract

In a model with matching frictions, we provide conditions under which repurchase agreements (or repos) co-exist with asset sales. In a repo, the seller agrees to repurchase the asset at a later date at the agreed price. Absent matching frictions, repos have no role. Introducing pairwise meetings, we show that agents prefer to sell asset whenever they face little uncertainty regarding the future use of the asset. As agents become more uncertain of the value of holding the asset, repos become more prevalent. We show that while the total volume of repos is always increasing with the uncertainty, the total sales volume is hump-shaped. In other words, pairwise matching alone is sufficient to explain why repo markets exist and there is no need to introduce information asymmetries or other market frictions.

1. Introduction

Many financial securities, including sovereign and corporate bonds, are traded on repo or securities lending markets where the seller agrees to repurchase the asset at a later date at

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*We thank Fernando Alvarez, Jennifer La’O, Randy Wright, and audiences at the Chicago Fed Money, Credit and Liquidity workshop, the Liquidity Conference in Madison, and at the European Central Bank for useful comments.
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a given price.¹ But why do these markets exist? In other words, while repo and securities lending markets coexist with market for ownership, this begs the questions: if agents have the means, why do they engage in repos, i.e. rent the asset, rather than just buying the asset to resale it later? Also, what is the impact of the repo market on asset sales? It is natural to expect that repo and sales volumes co-move negatively for a given set of traders, as these might be seen as two substitute activities.

Financial repos are usually associated with collateralized cash loans driven by the financing needs of the borrower. Therefore they appear at first sight only very remotely related – if at all – to the lending or renting of a security. However, in practice, repos are a key instruments for market participants in search of liquidity but also of a specific security. There are two ways a trader can acquire a security for a short term use: they can either engage in securities lending or conduct a “specials” repos. “Specials” are described in detail in Duffie (1996). Securities lending and “specials” have different legal and fiscal characterisctics, so that a trader may prefer one over the other, but their economic function is the same: they allow a trader to acquire a specific security temporarily.² The reasons why a trader needs to borrow a particular security vary, but generally the securities lent are needed to support a trading strategy or a settlement obligation. These motivations are further analyzed in CACEIS (2010), Duffie (1996) or Vayanos and Weill (2008), but for our purpose, it suffices to say that the security provides a service to the borrower that he values above and beyond its mere cash flows.³ To induce the lender of a security to trade, he usually obtains a lower repo rate than the prevailing money market rates, and invest the funds in money markets for a profit. The rights of the holder of a security acquired through a repo or securities lending are very similar: in a repo transaction, the buyer owns the collateral asset, he can re-use them during the term of the repo by selling the asset outright, “repoing” them or pledging them to a third party.⁴ In a securities-lending transaction, the borrower gains the ownership title to the securities lent while the lender gains full ownership of the title to the securities (or cash) pledged as collateral.⁵ Finally, both the repo and the securities lending markets

¹This is also true of many non-financial assets such as bicycles, cars, houses or airplanes, in which case we talk of leasing or renting.
²See CACEIS (2010) for a description of the different characteristics of repos and securities lending.
³Borrower may need to cover a failed transaction in the course of their trading activity, or a short position, or they may need to deliver securities they have not yet purchased against the exercise of a derivatives contract, or they want to raise specific collateral, perhaps for another securities lending transaction.
⁴See Monnet (2011) for the economics of rehypothecation.
⁵As explained in CACEIS (2010), the borrower can re-sell the securities borrowed, voting rights are transferred along with the title. Although the borrower, as owner of the securities, is entitled to the possible economic benefits associated with ownership such as dividends and coupons, he is under the contractual
involve trades negotiated bilaterally out of electronic trading platforms and their clearing is executed without the help of a central counterparty.\textsuperscript{6} Therefore, repo markets for the purpose of getting access to securities and the securities lending markets are very similar, and they look a lot like a market where borrowers are just renting the asset for a short period of time. From now on, we will refer to the securities lending market and the repos market as one same object, the repo market.

Some have argued that private information on the quality of assets can explain the existence of a repos market: The very fact that the seller is willing to repurchase the asset is a guarantee that the asset is of good quality (see, e.g. Koeppl and Chiu, 2011). However, this is hard to apply to Treasury securities or in a dynamic setting where agents learn the quality of the asset. Others have argued that repos are useful in order to cover counterparty risk, but there is then no difference between repos and collateralized loans (see, e.g. Mills and Reed). Finally, others have argued that selling an asset involves different costs than the one when conducting repos (see, e.g. Duffie 1996). In some sense, we would like to have a deeper understanding of the origin of these transaction costs, without necessarily resorting to different fiscal treatments or the inability of some agents to own some class of assets (such as market mutual funds).

In this paper, we show that repos can co-exist with assets sales even when the quality of the asset is known, when there is no risk exposure, and no differential fiscal treatments across types of trade. The essential elements are 1) agents trade in pair\textsuperscript{7} and 2) their personal valuation of the asset is uncertain. We assume that agents receive some preference shocks on the current utility from holding the asset, which may be more or less persistent. Once the shock hits, agents meet in pair. We concentrate on properties of allocations that are in the pairwise core, so that our result does not depend on the game played in each pairwise meeting. Once agents trade, they have to wait until the next trading session (say the next day) to change their position and they can only trade with one agent. This is the extent to which the trading frictions prevents the emergence of a Walrasian market outcome.

With pairwise matching, the analysis is complicated by the fact that agents’ asset holdings depend on their history of match. To simplify the analysis, we consider the case with two obligations to make equivalent payments in all distributions paid during the terms of the trade to the lender.


\textsuperscript{7}More generally, they are unable to trade within a group which aggregate valuation is identical to the Walrasian market valuation.
valuation shocks and we assume directed matching in the sense of Corbae, Temzelides and Wright (2003): Those agents who switched valuation are matched with each other. While this simplifies the analysis, it does not eliminate the distribution of asset holdings. We develop the model for random matching as well. With directed matching we show that an invariant distribution of assets has a two point support. The difference in these two points is increasing in the persistence of valuation shocks: As the probability to change valuation is increasing, agents tend to equate their asset holdings. Inversely, as the persistence increases, agents tend to hold different amount of the asset.

Interestingly, the total volume of asset sales is directly linked with the range of the support: As the difference in asset holdings increases, the sale of asset in a match is also increasing. This is intuitive: With directed matching, agents who just switched their valuation from high to low are matched with those agents who switched valuation from low to high. Therefore, as the difference in their asset holdings grow, also does the gains from trade, so that they trade a larger amount of the asset. However, since types are more persistent, fewer agents switch types so that the total volume of sales can either increase or decrease. We show that it is hump-shaped. Similarly, as the future valuation becomes uncertain, i.e. types are not persistent, agents are unwilling to change their position through asset sales, but they are willing to engage in repos. Therefore, the total volume of repos is decreasing with persistence and it is higher than total sales volume when the uncertainty is high (or persistence is low).

This has interesting implications for the organization of the repo market. In particular our theory predict that the repos market will be thinner when there is little uncertainty about one’s future preferences. Although we do not model it, we suspect that monetary policy (which is operated in the repo market) will have a higher impact then, as a lower quantity of repos can affect the market. Similarly, starting from a situation where agents know their future preferences, as uncertainty is growing, so is the volume of asset sales. Therefore, more sales have to be conducted in order to move the market. If we associate “normal times” with times when agents have a good idea about their future preferences, then monetary policy should be conducted with repos. However, with uncertainty growing overly large, monetary policy will be more effective in moving market if it is conducted via asset sales/purchases.

The paper proceeds as follows. In Section 2 we describe the model. In Section 3 we define and characterize pairwise core allocations. In Section 4, we solve for the equilibrium when there is random matching, in the extreme cases when there is full persistence of the preference
shocks and no persistence at all. Section 5 analyzes the case with directed matching and solve for the equilibrium distribution and volumes in general. Section 6 presents an example with Nash bargaining rather than pairwise core allocations.

2. The Model

This is a version of Koeppl, Monnet, and Temzelides (2008). Time is discrete and the horizon is infinite. Each period has two sub-periods: A trading stage, followed by a settlement stage. There is a continuum of agents. In each period, there is a measure $1/2$ of two types of agents, type $h$ and type $\ell$. The type of an agent switches randomly and with probability $1-\pi \in [1/2, 1]$ at the start of the transaction stage. The law of large numbers then guarantees that there is the same measure of each type in each period. Agents are anonymous in the trading stage and their type is private information. A clearinghouse records the transactions of an agent.

There is a long-lived asset in fixed supply $A$. As in Lagos and Rocheteau (2009), we associate this asset to a Lucas-tree: One unit of the asset yields one unit of some fruit in the settlement stage. Agents of type $i \in \{h, \ell\}$ derive utility $u_i(a)$ from holding $a$ units of the asset.\footnote{There are several interpretations for this formulation: Lagos and Rocheteau argue that this is the utility derived from the tree’s fruit. Duffie, Garleanu and Pedersen (2009) explain that these are preferences from liquidity, hedging or other benefits that holding the assets may yield.} For simplicity, we impose the following condition,

**Assumption 1.** $u'_h(a) \geq u'_\ell(a)$ for all $a$.

Therefore, for a given level of asset holding, the agent with the high type has a higher marginal utility than the agent with the low type. To be concise, we will refer to agents of type $h$ as agents $h$ and to agents of type $\ell$ as agents $\ell$.

In the trading stage, agents can agree to trade the asset, in which case the seller transfers the assets and the fruits in the settlement stage. Or agents can only agree to trade the fruit of the asset: Then the seller only transfers the fruits that it yields, while he maintains ownership over the asset. We interpret this second trade as a repo trade, as the buyer surrenders the asset back to the seller once he enjoyed the benefits of holding it this period.

While the trading stage can be seen as a market, there is no market in the settlement stage. In the settlement stage, agents are endowed with a production technology for the
numeraire good $m$. It costs them one unit of disutility to produce one unit of this good so that the numeraire good is akin to transferable utility. Agents also consume the numeraire good and derive one unit of utility for each unit they consume. If utility is transferable, the settlement stage does not generate any net utility gains. The numeraire good will be the settlement asset. In the settlement stage, agents settle the terms of the trade that were agreed upon in the previous trading stage.

3. Benchmark Walrasian Market

We first consider the case where the trading stage is a Walrasian market. A repo trades at price $p^r$ while the asset sales at price $p$. We consider only stationary equilibrium so that these prices are the same in each period. An agent $i = h, \ell$ with asset holdings $a$ has a value $W_i(a)$ of holding the asset, where $W_i(a)$ is defined recursively as

$$W_i(a) = \max_{a_i, q_i^r} u_i(a_i + q_i^r) - d + \beta E_{k|i} W_k(a_i)$$

s.t. $d + pa = pa_i + p^r q_i^r$

where the agent repos $q_i^r$ and purchases an amount $a_i$ of the asset. Naturally the quantity of repos $q_i^r$ does not enter in the continuation valuation but only in the momentary utility $u_i(.)$. The first order and envelope conditions yield

$$u_i'(a_i + q_i^r) + \beta E_{k|i} W_k'(a_i) = p$$

$$u_i'(a_i + q_i^r) = p^r$$

$$W_i'(a) = p$$

Notice that all agents value an additional unit of the asset in the same way when they enter the Walrasian market, independent of their type or of their asset holdings. There are two reasons for this: First, the utility is linear in the numeraire good such that there is no wealth effect in this model and, second, agents are playing against the whole market. In the next section, we will modify the latter. For the time being, the equilibrium prices and
quantities satisfy,

\[(1 - \beta)p = \rho^r\]
\[u'_h(a_h + q_h^r) = \rho^r\]
\[u'_\ell(a_\ell + q_\ell^r) = \rho^r\]
\[(a_h + q_h^r) + (a_\ell + q_\ell^r) = 2A\]

The first equation is a no-arbitrage condition: Agents have to be indifferent between conducting a repo, in which case they have to pay the price \(\rho^r\) in terms of the numeraire good, and buying the asset at price \(p\) and reselling it in the next period at price \(\beta p\). These two schemes are payoff equivalent and so should be their cost. As a consequence, anything goes for repos, and in particular \(q_h^r = q_\ell^r = 0\). In other words, in a Walrasian market, there is no difference between conducting a repos or buying and selling the asset. Therefore, absent any additional frictions, the Walrasian benchmark is not helpful to study the structure of the repo and other rental markets. In the following section, we depart from the Walrasian benchmark by assuming that agents can only meet in pair.

4. **Pairwise core allocations**

We now assume that each agent \(h\) is matched with exactly one agent \(\ell\) in the trading stage. We consider allocations in the pairwise core. To define an allocation, we will consider a generic meeting between an agent \(h\) holding a generic amount of the asset \(a_h\) and an agent \(\ell\) holding a generic amount of the asset \(a_\ell\). An allocation is a triple \(\{q^s(a_h, a_\ell), q^r(a_h, a_\ell), d(a_h, a_\ell)\}\) where \(q^s\) denotes the quantity of the asset that the agent \(h\) buys from the agent \(\ell\) (sells if negative), \(q^r\) is the quantity of the asset that the agent \(h\) buys or repo from the agent \(\ell\) (sells or reverse repo if negative) and \(d\) is the numeraire transfer that the agent \(h\) makes in the settlement stage to the agent \(\ell\) (receives if negative). We only focus on stationary and symmetric allocations. An allocation is feasible if

\[q^s(a_h, a_\ell) \in [-a_h, a_\ell]\]
\[q^r(a_h, a_\ell) + q^s(a_h, a_\ell) \in [-a_h, a_\ell]\]
We will denote by \((q^s, q^r, d)\) the feasible allocations for all possible matches such that \((q^s, q^r, d)\) defines invariant distributions of asset holdings for agents \(h\) and \(\ell\). We denote these distributions by \(\mu_i(a)\) for \(i \in \{h, \ell\}\), where we have dropped the reference to the allocation for convenience. If they exist, a property of any invariant distribution is that

\[
\frac{1}{2} \int ad\mu_h(a) + \frac{1}{2} \int ad\mu_\ell(a) = A
\]

Then we can define recursively the expected value for agent \(i \in \{h, \ell\}\) of holding asset \(a\), before entering the trading stage, \(V_i(a)\), as

\[
V_h(a) = \pi \int [u_h(a + q^s(a, a_\ell) + q^r(a, a_\ell)) - d(a, a_\ell) + \beta V_h(a + q^s(a, a_\ell))]d\mu_\ell(a_\ell) \\
+ \ (1 - \pi) \int [u_\ell(a - q^s(a_h, a) - q^r(a_h, a)) + d(a_h, a) + \beta V_\ell(a - q^s(a_h, a))]d\mu_h(a_h)
\]

With probability \(\pi\) the agent \(h\) remains an agent \(h\). Then he meets an agent \(\ell\) with asset \(a_\ell\) according to the distribution \(\mu_\ell\). Since he remains an agent \(h\), he enjoys instant utility \(u_h(.\) from his asset holdings \(a + q^s(a, a_\ell) + q^r(a, a_\ell)\) at the end of the settlement stage. However, he only carries \(a + q^s(a, a_\ell)\) over to the next period since repos do not involve the transfer of the asset but only of fruits. The agent values this portfolio according to \(\beta V_h(a + q^s(a, a_\ell))\).

With probability \(1 - \pi\) the agent \(h\) becomes an agent \(\ell\). In this case, he meets an agent \(h\) according to the distribution \(\mu_h\) and he enjoys instant utility \(u_\ell(.\) from his asset holdings \(a - q^s(a_h, a) - q^r(a_h, a)\). He values his remaining portfolio according to \(\beta V_\ell(a - q^s(a_h, a))\). Similarly for agents \(\ell\),

\[
V_\ell(a) = \pi \int [u_\ell(a - q^s(a_h, a) - q^r(a_h, a)) + d(a_h, a) + \beta V_\ell(a - q^s(a_h, a))]d\mu_h(a_h) \\
+ \ (1 - \pi) \int [u_h(a + q^s(a, a_\ell) + q^r(a, a_\ell)) - d(a, a_\ell) + \beta V_h(a + q^s(a, a_\ell))]d\mu_\ell(a_\ell)
\]

We assume that there is limited commitment, and an allocation \((q^s, q^r, d)\) is individually rational if all agents prefer the allocation to being in autarky this period. That is, for any portfolio \(a\), an agent \(h\) matched with an agent \(\ell\) with a portfolio \(a_\ell\) prefers the allocation than
not trading today, i.e.

\[ u_h(a + q^s(a, a_\ell) + q^r(a, a_\ell)) - d(a, a_\ell) + \beta V_h(a + q^s(a, a_\ell)) \geq u_h(a) + \beta V_h(a), \]

and similarly for an agent \( \ell \) matched with an agent \( h \) with portfolio \( a_h \),

\[ u_\ell(a - q^s(a_h, a) - q^r(a_h, a)) + d(a_h, a) + \beta V_\ell(a - q^s(a_h, a)) \geq u_\ell(a) + \beta V_\ell(a). \]

From now on, for concision and whenever there is no risk of confusion, we will drop references to the agents’ portfolios in an allocation. We are interested in allocations from which no agent in a match or the pair of agents, has an interest in deviating. That is, as in Zhu, Wallace and Kennan (2009) and Rocheteau (2011), we concentrate on allocations that are in the pairwise core. Given an increasing and concave utility function \( U(.) \), we define the pairwise core in a match where agent \( h \) and \( \ell \) hold portfolio \( a_h \) and \( a_\ell \) respectively, as the set of allocations that satisfy

\[
(q^s, q^r, d) = \text{argmax} [u_h(a_h + q^s + q^r) - d + \beta V_h(a_h + q^s)] \\
s.t. \quad q^s \in [-a_h, a_\ell], \; q^s + q^r \in [-a_h, a_\ell] \\
\quad \quad u_\ell(a_\ell - q^s - q^r) + d + \beta V_\ell(a_\ell - q^s) \geq U(a_\ell) \quad (3) \\
\quad \quad u_h(a_h + q^s + q^r) - d + \beta V_h(a_h + q^s) \geq u_h(a_h) + \beta V_h(a_h)
\]

The first set of constraint is the two feasibility constraints. The second constraint is the participation constraint for type 2 agents, while the first constraint is the participation constraint for type 1 agents. While we do not take a stance on what the function \( U(.) \) is, it has to satisfies some simple properties. For example, to insure the participation of agents \( \ell \), it must be that \( U(a_\ell) \geq u_\ell(a_\ell) + \beta V_\ell(a_\ell) \) for all \( a_\ell \). Also, given \( a_\ell \), \( U(a_\ell) \) should be increasing in \( \beta \). To fix ideas, we could set \( U(a_\ell) = \lambda[u_\ell(a_\ell) + \beta V_\ell(a_\ell)] \) where \( \lambda \geq 1 \). Finally, notice that the participation constraint of type 2 agents is always binding: Otherwise, it would be possible to lower \( d \) and raise the utility of the type 1 agent.

Given a pair \((a_h, a_\ell)\) a pairwise core allocation with \( q^s \in (-a_h, a_\ell) \) satisfies the following
first order conditions,

\[ q^\ast : (1 + \mu)[u'_h(a_h + q^s + q^r) + \beta V'_h(a_h + q^s)] \]
\[-\lambda[u'_\ell(a_\ell - q^s - q^r) + \beta V'_\ell(a_\ell - q^s)] + \xi_h - \xi_\ell = 0 \]
\[ q^r : (1 + \mu)u'_h(a_h + q^s + q^r) - \lambda u'_\ell(a_\ell - q^s - q^r) + \xi_h - \xi_\ell = 0 \]
\[ d : -1 + \lambda - \mu = 0 \]

These three conditions give us\(^9\)

\[ V'_h(a_h + q^s) = V'_\ell(a_\ell - q^s) \] \hspace{1cm} (4)

and

\[ u'_h(a_h + q^s + q^r) = u'_\ell(a_\ell - q^s - q^r) \quad \text{if} \quad \xi_i = 0, i = h, \ell \] \hspace{1cm} (5)
\[ q^s + q^r = a_\ell \quad \text{if} \quad \xi_\ell > 0 \]
\[ q^s + q^r = -a_h \quad \text{if} \quad \xi_h > 0 \]

Equations (4) and (5) characterize the pairwise core allocations \(q^\ast(a_h, a_\ell)\) and \(q^r(a_h, a_\ell)\), while \(d(a_h, a_\ell)\) is given by (3) holding with equality. Notice that the allocation depends on the distributions of asset holdings \(\mu_i\) for \(i = h, \ell\) as they affect the value functions \(V_i\). To fully characterize the equilibrium with an invariant distribution, we need to specify how agents are matched. In the next section, we assume that agents are randomly matched. Then we assume that agents are matched according to a pre-specified matching rule.

5. Random Matching: Special Cases

Here, we study two extreme cases with either \(\pi = 1/2\) or \(\pi = 1\) and an agent \(h\) is randomly matched with an agent \(\ell\). In the case with \(\pi = 1/2\), types have no persistence and each types are as likely for an agent independently of his history of type. In the case with \(\pi = 1\) types are fully persistent as types are fixed for ever.

With no persistence and random matching, we obtain the following result.

**Proposition 2.** With random matching and \(\pi = 1/2\), the pairwise core allocations defines

\(^9\)In the case where \(q^\ast = a_\ell\), (4) becomes \(V'_h(a_h + q^s) > V'_\ell(a_\ell - q^s)\).
a unique equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a} = A/2$ with $q^s(\bar{a}, \bar{a}) = 0$, and $q^r(\bar{a}, \bar{a}) > 0$.

In the case without persistence, (1) and (2) imply that $V_h(a) = V_{\ell}(a)$ for all $a$, such that agents $h$ and $\ell$ value future payoff of holding the asset in the same way. In this case, (4) implies that $a_h + q^s(a_h, a_\ell) = a_\ell - q^s(a_h, a_\ell)$ with $q^s(a_h, a_\ell) > 0$ if and only if $a_\ell > a_h$ and $q^s(a_h, a_\ell) < 0$ otherwise. Therefore, the unique equilibrium is one where the distribution of asset holding is degenerate at $\bar{a} = A$ and $q^s(\bar{a}, \bar{a}) = 0$. This is very intuitive: Since all agents give the same value to future returns, they extinguish all surplus from trading the asset by averaging their asset holding (i.e. once an agent holding $a_h$ trade with an agent holding $a_\ell$, they both end up with $(a_h + a_\ell)/2$ and in equilibrium they hold the same amount of the asset. Then (5) together with assumption 1 imply that $q^r(\bar{a}, \bar{a}) > 0$: While agents value future asset returns the same way, they differ in their valuation of current return. Therefore, there is a benefit from repos, where only the current return is traded.

With full persistence, the result is that there is neither asset sales nor repo in equilibrium.

**Proposition 3.** With random matching and $\pi = 1$, the pairwise core allocations defines an equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a}_h$ and $\bar{a}_\ell$ with $\bar{a}_h > \bar{a}_\ell$ where $q^s(\bar{a}_h, \bar{a}_\ell) = 0$ and $q^r(\bar{a}_h, \bar{a}_\ell) = 0$.

We will first verify that the proposed allocation is an equilibrium. Since $q^s(\bar{a}_h, \bar{a}_\ell) = 0$ and $q^r(\bar{a}_h, \bar{a}_\ell) = 0$, the allocation must be such that $d(\bar{a}_h, \bar{a}_\ell) = 0$ as agents would otherwise prefer autarky. Using (1) and (2), we then have for $i = h, \ell$,

$$V_i(\bar{a}_i) = \frac{u_i(\bar{a}_i)}{1 - \beta}$$

and (4) and (5) imply that $\bar{a}_h$ and $\bar{a}_\ell$ are uniquely given by

$$u'_h(\bar{a}_h) = u'_\ell(\bar{a}_\ell)$$

with $\bar{a}_h = 2A - \bar{a}_\ell$. This verifies that there is no asset sales or repos in equilibrium.\textsuperscript{10} Also, this verifies that $\bar{a}_h > \bar{a}_\ell$. This equilibrium is unique whenever endowments are symmetric across all agents (and no constraint bind – which may happen if some agents $\ell$ are endowed with too many securities in the first place) so that all agents $\ell$ hold the same amount $a_\ell$ and

\textsuperscript{10}Notice that $\bar{a}_\ell = 0$ is only an equilibrium if $\bar{a}_h = 0$ was agents $\ell$ initial endowment (since they would prefer autarky otherwise) and $U(\bar{a}_\ell) = 0$. 

all agents $h$ holds $a_h$. To see this notice that if an agent $h$ endowed with $a_h$ meets an agent $\ell$ endowed with $a_\ell$, then the pairwise core dictates that they trade so that (5) holds. But the unique solution is that $a_h + q^s(a_h, a_\ell) = a_h$ and $a_\ell - q^s(a_h, a_\ell) = a_\ell$. Since $a_h + a_\ell = 2A$, such a $q^s$ exists and takes the agents directly to the equilibrium distribution of asset holdings.

For general levels of persistence $\pi \in (0, 1)$ and random matching, we are unable to determine analytically the total volume of sales and repos as we cannot solve analytically for the invariant equilibrium distribution of asset holdings. However, we can partially characterize sales and repos within one match.

**Proposition 4.** With random matching and $\pi \in (0, 1)$, if $V_i$ is concave and differentiable, any pairwise core allocation is such that $q^s(a_h, a_\ell)$ is increasing in $a_\ell$ and decreasing in $a_h$. Also $q^s(a_h, a_\ell) + q^r(a_h, a_\ell)$ is increasing in $a_\ell$ and decreasing in $a_h$.

We leave the proof in the Appendix. Proposition 4 gives us some indication of how the distribution of asset holding moves in equilibrium. As an agent $\ell$ is better endowed, he will sell more to agent $h$, and as agent $h$ is less endowed, he will buy more from agent $\ell$. This hints to more trade as agents valuation differ and we expect that the distributions of asset holdings become more spiked around their respective mean $a_h$ and $a_\ell$ as $\pi$ increases, where the means are diverging as $\pi$ increases. However, since agents can switch randomly from one type to the other, it is difficult to fully characterize the equilibrium. Also, in the next section we show that this claim is true when matching is directed.

6. Directed search

We now describe the matching technology. Following Corbae, Temzelides and Wright (2003), we use directed search and the matching function specifies that those agents who just switched to new types meet with each other. Below we verify that matching function is an equilibrium matching rule (where such a term is precisely defined). The rest of this section is devoted to proving the following result.

**Proposition 5.** With directed search, the pairwise core allocations defines an equilibrium characterized by a distribution of asset holdings for each type that are degenerate at some level $\bar{a}_i$ with $i = h, \ell$ with $q^s(\bar{a}_h, \bar{a}_\ell) = 0$, $q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell$ and $q^r(\bar{a}_h, \bar{a}_\ell) = q^r(\bar{a}_\ell, \bar{a}_h) = q^r$ where $q^r$ solves $u_h'(\bar{a}_h + q^r) \geq u_\ell'(\bar{a}_\ell - q^r)$ (with equality if $q^r < \bar{a}_\ell$).

We proceed by guessing and verifying that the two distributions of assets are degenerate at $\bar{a}_h$ and $\bar{a}_\ell$ for high and low types respectively. As agents who did not switch are matched
together and the distribution of asset holdings is invariant, the pairwise core allocation is
\( q^s(\bar{a}_h, \bar{a}_\ell) = 0 \) and \( \bar{a}_h \) and \( \bar{a}_\ell \) satisfy
\[
V'_h(\bar{a}_h) = V'_\ell(\bar{a}_\ell)
\]
with \( q^r(\bar{a}_h, \bar{a}_\ell) \) given by
\[
u'_h(\bar{a}_h + q^r) = u'_\ell(\bar{a}_\ell - q^r) \quad \text{if} \quad \xi_i = 0, i = 1, 2 \quad (7)
\]
\[
q^r = \bar{a}_\ell \quad \text{if} \quad \xi_2 > 0 \quad (8)
\]
For those agents who switched we have that a "new" agent \( \ell \) is holding \( \bar{a}_h \) while a "new" agent \( h \) is holding \( \bar{a}_\ell \). Therefore, the pairwise core allocation for those agents who switched is given by
\[
V'_h(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h)) = V'_\ell(\bar{a}_h - q^s(\bar{a}_\ell, \bar{a}_h))
\]
so that
\[
q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell \quad (9)
\]
We will guess and later verify that \( q^s \geq 0 \). A repo trade satisfies
\[
u'_h(\bar{a}_\ell + q^s + q^r) = u'_\ell(\bar{a}_h - q^s - q^r) \quad \text{if} \quad \xi_i = 0, i = 1, 2 \quad (10)
\]
\[
q^s + q^r = \bar{a}_h \quad \text{if} \quad \xi_\ell > 0 \quad (11)
\]
Hence, using (9) to replace \( q^s(\bar{a}_\ell, \bar{a}_h) \) in (10) and (11) and comparing the result with (7) and (8), we obtain
\[
q^r(\bar{a}_\ell, \bar{a}_h) = q^r(\bar{a}_h, \bar{a}_\ell).
\]
In words, the property of any core allocations with direct matching is that agents who just
switched type adjust their asset holdings so that they hold their type’s portfolio. Then they
conduct repo as if they never switched. Agents who did not switch type just engage in
repo. Loosely speaking, there is a sense in which agents first access the asset market and
then engage in repo. To verify that this is an equilibrium we need to verify that an agent
would not prefer to be matched with a different agent than the one he is assigned to, or
that no agent would prefer to interact with him to trading with his assigned agent. In the
terminology of Corbae, Temzelides and Wright (2003), the proposed matching rule is an
equilibrium matching if no coalition consisting of 1 or 2 agents can do better (in the sense
that \( u_i(q^*, q^r, d) + \beta V_i(q^*, q^r, d) \) increases for all \( i \) in the coalition) by deviating in the following sense: An individual can deviate by matching with himself (i.e. being in autarky this period) rather than as prescribed by the matching rule; and a pair can deviate by matching with each other rather than as prescribed by the matching rule.

It should be clear that pairwise core allocations are always better than autarky. Therefore we only need to check deviations by a coalition of 2 agents. However, we can rule out deviations that involve agents \( a \) as they always obtain utility \( U(a) \) whenever they hold assets \( a \) (be it on or off equilibrium) and so they are unable to do better by deviating. The only relevant deviation we need to check is one where an agent \( h \) holding \( \bar{a}_h \) (an agent \( h \) who did not switch) is matched with an agent \( \ell \) holding \( \bar{a}_\ell \) (an agent who was \( a \) and just switched). However, both agents like to hold the asset this period and so the agent \( h \) holding \( \bar{a}_h \) is indifferent trading with an agent \( \ell \) holding \( \bar{a}_\ell \) or an agent \( h \) holding \( \bar{a}_\ell \) as the allocation would give him the same payoff:

\[
\begin{align*}
    u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) \\
    = \bar{u}_h - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) \\
    = \bar{u}_h + \bar{u}_\ell + \beta V_\ell(\bar{a}_\ell) + \beta V_h(\bar{a}_h) - U(\bar{a}_\ell) \geq u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)
\end{align*}
\]

where \( \bar{u}_h = u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) \) and \( \bar{u}_\ell = u_\ell(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell)) \). However, the agent \( h \) holding \( \bar{a}_\ell \) would have to give at least \( u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \) to the other agent \( h \) and he is worse off if \( u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h) \). We show that this is the case in the Appendix, so that an agent \( h \) holding \( \bar{a}_\ell \) would never want to meet an agent \( h \) with \( \bar{a}_h \). Therefore, the proposed directed matching rule together with the pairwise core allocation define an equilibrium where the distribution of asset holding for each type is degenerate. The following proposition characterizes the support of the distributions.

**Proposition 6.** The degenerate supports \( \bar{a}_h \) and \( \bar{a}_\ell \) of the two distributions are fully characterized by the following equations,

\[
\begin{align*}
    \bar{a}_h + \bar{a}_\ell &= 2A \tag{12} \\
    u_h'(\bar{a}_h + q^r) &= u_\ell'(\bar{a}_\ell - q^r) \tag{13} \\
    u_h'(\bar{a}_h + q^r) &= \frac{1}{(2\pi - 1)} \left\{(1 - \beta(1 - \pi))U'(\bar{a}_\ell) - (1 - \beta(1 - \pi))(1 - \pi)U'(\bar{a}_h)\right\} \tag{14}
\end{align*}
\]

The proof is in the Appendix. There we also show that the equilibrium asset holdings
are diverging in the persistence of the shock, i.e. as $\pi$ increases.

**Corollary 7.** $\tilde{a}_h - \tilde{a}_\ell \geq 0$ is increasing in $\pi$. Sales volume is hump-shaped in $\pi$ while repos volume is strictly decreasing in $\pi$.

The intuition for this result is straightforward. When $\pi = 1/2$, agents future type is independent of their current type. Therefore, two agents’ future value of the asset is the same. Since the core pairwise allocation equates the marginal benefit of holding the asset, all agents hold the same quantity of assets. Therefore, $\tilde{a}_h(1/2) = \tilde{a}_\ell(1/2) = A$ and there are no asset sales, but only repos that allocate the fruits to those agents $h$ who like it most.\(^{11}\) As $\pi$ increases, it is more likely that an agent $h$ becomes once again an agent $h$ next period. Therefore his valuation for the asset increases, and starting from $a_h = a_\ell$, there are gains from trade when an agent $h$ meets an agent $\ell$. In this case, $\tilde{a}_h > A > \tilde{a}_\ell$. In equilibrium only those agents who switch types have gains to trade the asset and so the volume of asset sales is increasing. Also repos are decreasing as agents hold more of the asset they like. Finally, when $\pi = 1$, agents know their type for sure. Hence in equilibrium, all gains from trades (be it asset trade or fruit trade) are extinguished, so that there is neither sales nor repos.

Directed search is very powerful in the sense that there are many allocations that are payoff equivalent to the distribution we just described. However, not all allocations belong to the core. For example, consider the case with full persistence where $\pi = 1$. It is easy to see that we can only use repos to achieve the same payoff as the allocation described above. Here is how: Let us consider the case where the core allocation $(\tilde{a}_h, \tilde{a}_\ell)$ as defined in Proposition 6 is such that $\tilde{a}_\ell < A < \tilde{a}_h$. Let us endow all agents with $A$. Then match each agent $h$ with an agent $\ell$. By specifying $q^* = 0$ and $\bar{q}^* \leq A$ such that $A + \bar{q}^* = \tilde{a}_h + q^*$, we obviously satisfy the intratemporal condition for a core allocation $u'_{h}(A + \bar{q}^*) = u'_{\ell}(A - \bar{q}^*)$, and we achieve the same payoff as the allocation in Proposition 6. However, interestingly, it does not necessarily satisfy the intertemporal condition for a core allocation. Indeed, suppose it is in the core. Then we must have $V''_{h}(A) = V''_{\ell}(A)$. Then with $\pi = 1$, (14) becomes

\[
\begin{align*}
    u'_{h}(A + \bar{q}^*) &= (1 - \beta)U'(A) \\
    \text{and replacing } A + \bar{q}^* &= \tilde{a}_h + q^* \text{ we obtain } u'_{h}(\tilde{a}_h + q^*) &= (1 - \beta)U'(A). \quad \text{Since } \tilde{a}_\ell < A \text{ we have } U'(A) < U'(\tilde{a}_\ell) \text{ so that } u'_{h}(\tilde{a}_h + q^*) < U'(\tilde{a}_\ell) \text{ which contradicts the fact that our original allocation was a core allocation. Therefore, we cannot replicate the allocation in Proposition}
\end{align*}
\]\(^{11}\)

Note that (14) is always satisfied at $\pi = 1/2$ and $\tilde{a}_h(1/2) = \tilde{a}_\ell(1/2) = A$ since cross multiplying by $2\pi - 1$ we obtain that both sides are zero.
With \( a_i = A \) and with repo only: We need the distribution of assets to be different across types.

### 7. Directed Search With Nash Bargaining

In this section, we consider that the allocation is set by Nash bargaining rather than considering an arbitrary pairwise core allocation. We still assume that agents who did not switch are matched together while those agents who just switched are matched with each other. With Nash bargaining, the allocation of an agent \( h \) with portfolio \( a_h \) matched with an agent \( \ell \) with portfolio \( a_\ell \) solves the following problem:

\[
\max_{q^s, q^r, d} \left[ u_h(a_h + q^s) - d + \beta V_h(a_h + q^s) - u_h(a_h) - \beta V_h(a_h) \right]^\theta \\
\times \left[ u_\ell(a_\ell - q^r) + d + \beta V_\ell(a_\ell - q^r) - u_\ell(a_\ell) - \beta V_\ell(a_\ell) \right]^{1-\theta}
\]

with first order conditions

\[
V'_h(a_h + q^s) = V'_\ell(a_\ell - q^r) \\
u'_h(a_h + q^s + q^r) = u'_\ell(a_h - q^s - q^r) \\
d(a_h, a_\ell) = (1 - \theta)[u_h(a_h + q^s + q^r) - u_h(a_h) + \beta V_h(a_h + q^s) - \beta V_h(a_h)] \\
-\theta[u_\ell(a_\ell - q^s - q^r) - u_\ell(a_\ell) + \beta V_\ell(a_\ell - q^s) - \beta V_\ell(a_\ell)]
\]

We still assume that agents who did not switch types are matched together while those agents who just switched are matched together. In the Appendix, we show

**Proposition 8.** With directed search, an equilibrium with bargaining is characterized by a distribution of asset holdings for each type that are degenerate at some level \( \bar{a}_i \) with \( i = h, \ell \) with \( q^s(\bar{a}_h, \bar{a}_\ell) = 0 \), \( q^s(\bar{a}_h, \bar{a}_\ell) = \bar{a}_h - \bar{a}_\ell \) and \( q^r(\bar{a}_h, \bar{a}_\ell) = q^r(\bar{a}_h, \bar{a}_\ell) = q^r \) where \( q^r \) solves \( u'_h(\bar{a}_h + q^r) \geq u'_\ell(\bar{a}_\ell - q^r) \) (with equality if \( q^r < \bar{a}_\ell \)).

Notice that \( d(\bar{a}_h, \bar{a}_\ell) \) is the price of repos while \( d(\bar{a}_\ell, \bar{a}_h) \) is the price of an asset sale and a repo, so that we should expect \( d(\bar{a}_h, \bar{a}_\ell) > d(\bar{a}_\ell, \bar{a}_h) \). In the Appendix, we show that

\[
d(\bar{a}_h, \bar{a}_\ell) = (1 - \theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell) - u_\ell(\bar{a}_\ell - q^r)] \\
d(\bar{a}_\ell, \bar{a}_h) = d(\bar{a}_h, \bar{a}_\ell) + \bar{u} + \beta(1 - \theta)[V_h(\bar{a}_h) - V_h(\bar{a}_\ell)] + \beta \theta[V_\ell(\bar{a}_h) - V_\ell(\bar{a}_\ell)]
\]
where
\[ \bar{u} = (1 - \theta)[u_h(\bar{a}_h) - u_h(\bar{a}_\ell)] + \theta[u_\ell(\bar{a}_h) - u_\ell(\bar{a}_\ell)] \]

We consider the following matching technology: An agent \( h \) with \( \bar{a}_h \) meets an agent \( \ell \) with \( \bar{a}_\ell \) and an agent \( h \) with \( \bar{a}_h \) meets an agent \( \ell \) with \( \bar{a}_\ell \). Given \( q^s(\bar{a}_h, \bar{a}_\ell) = 0 \), \( q^s(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell \) and \( q^r(\bar{a}_h, \bar{a}_\ell) = q^r(\bar{a}_\ell, \bar{a}_h) = q^r \), we obtain the following value functions,

\[ V_h(\bar{a}_h) = \pi[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] + (1 - \pi)[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] \]
\[ V_\ell(\bar{a}_\ell) = \pi[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi)[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] \]

Notice that we need to specify the value of the outside option in order to solve for the bargaining solution in equilibrium, i.e. \( V_h(\bar{a}_\ell) \) and \( V_\ell(\bar{a}_h) \). A moment reflection should convince the reader that, given our matching technology,

\[ V_h(\bar{a}_\ell) = \pi[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] + (1 - \pi)[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] \]
\[ V_\ell(\bar{a}_h) = \pi[u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi)[u_h(\bar{a}_h + q^r) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] \]

Using these value functions we find that
\[ d(\bar{a}_\ell, \bar{a}_h) = d(\bar{a}_h, \bar{a}_\ell) + \frac{\bar{u}}{1 - \beta} \]
so that the value of selling the asset is just \( \bar{u}/(1 - \beta) \), the lifetime discounted surplus from adjusting portfolios. Notice that as \( \pi \to 1/2 \) we have \( \bar{a}_h \to \bar{a}_\ell \) so that \( \bar{u} \to 0 \) and there is no value of selling the asset. Figures 1 and 2 illustrate this equilibrium. This is illustrated in the two following figures.

Figure 1 shows the indifference curves for \( V_i(a) + d \) for two agents: One agent \( h \) endowed with \( a_\ell \) and one agent \( \ell \) endowed with \( a_h \). Indifference curves are tangent at the stationary distribution points \( (\bar{a}_h, \bar{a}_\ell) \) (the axis for \( d \) is reversed). Scaling the continuation utility by \( 1 - \beta \) to compare it with present utility, notice that \( (1 - \beta)V_h'(a) \leq u'_h(a) \) by assumption 1 and since there is chance that an agent \( h \) reverts to being an agent \( \ell \) in the future. By the same argument, notice that \( (1 - \beta)V_\ell'(a) \leq u'_\ell(a) \). Therefore, as illustrated in Figure 2, the indifference curves for \( u_i(a) + d \) for both agents will be tangent at a point south-east of the \( (\bar{a}_h, \bar{a}_\ell) \). This explains why repos are useful: They exploit intratemporal gains from trade.
Figure 1: Intertemporal gains from trade

Figure 2: Intratemporal gains from trade
In the Appendix, we solve for the solution to the bargaining problem with directed matching \( q^r, \bar{a}_h \) and \( \bar{a}_\ell \).

**Proposition 9.** The degenerate supports \( \bar{a}_h \) and \( \bar{a}_\ell \) of the two distributions with bargaining are fully characterized by the following equations,

\[
\begin{align*}
    u'_h(\bar{a}_h + q^r) &= u'_\ell(\bar{a}_\ell - q^r) \\
    u'_h(\bar{a}_h + q^r) &= \frac{[\pi - (2\pi - 1)\beta][\theta u'_\ell(\bar{a}_\ell) - (1 - \theta)u'_h(\bar{a}_h)] - (1 - \pi)[\theta u'_\ell(\bar{a}_h) - (1 - \theta)u'_h(\bar{a}_\ell)]}{(2\pi - 1)(1 - \beta)(2\theta - 1)} \\
    \bar{a}_h + \bar{a}_\ell &= 2A
\end{align*}
\]

Notice from the second equation that when \( \pi = 1/2 \), we must have

\[
\theta[u'_\ell(\bar{a}_\ell) - u'_h(\bar{a}_h)] = (1 - \theta)[u'_h(\bar{a}_h) - u'_h(\bar{a}_\ell)]
\]

and the unique solution is \( \bar{a}_\ell = \bar{a}_h = A \). In this case, \( q^r = 0 \) and \( q^r > 0 \). Also, if \( \pi = 1 \) we have

\[
u'_h(\bar{a}_h + q^r) = \frac{\theta u'_\ell(\bar{a}_\ell) - (1 - \theta)u'_h(\bar{a}_h)}{(2\theta - 1)}
\]

with solution \( q^r = 0 \) and \( u'_\ell(\bar{a}_\ell) = u'_h(\bar{a}_h) \). In this case as well, \( q^r = 0 \). We can also find the price for repo and asset sales. This is our final result.

**Corollary 10.** Let \( p^r \) be the price of a repo and \( p^s \) the price of a sale. Then

\[
\begin{align*}
p^r &= \frac{(1 - \theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell) - u_\ell(\bar{a}_\ell - q^r)]}{q^r} \\
p^s &= \frac{(1 - \theta)[u_h(\bar{a}_h) - u_h(\bar{a}_\ell)] + \theta[u_\ell(\bar{a}_h) - u_\ell(\bar{a}_\ell)]}{(1 - \beta)(\bar{a}_h - \bar{a}_\ell)}
\end{align*}
\]

From the transfers \( d(a_h, a_\ell) \) we have \( d(\bar{a}_h, \bar{a}_\ell) = p^r q^r \) as the pair of agents who did not switch types only conduct repos. Therefore,

\[
p^r q^r = (1 - \theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell) - u_\ell(\bar{a}_\ell - q^r)].
\]

Also, since the pair of agents that switched conducts both an asset sale \( q^s \) to adjust their
position, and then a repo. Therefore \( d(\bar{a}_t, \bar{a}_h) = p^r q^r + p^s q^s \). Since

\[
d(\bar{a}_t, \bar{a}_h) = d(\bar{a}_h, \bar{a}_t) + \frac{\bar{u}}{1 - \beta}
\]

we obtain that

\[
p^s q^s = \frac{\bar{u}}{1 - \beta}
\]

and using the expression for \( \bar{u} \), with \( q^s = \bar{a}_h - \bar{a}_t \), we get the result.

It is clear that in general those prices are different from their Walrasian equivalent, and in particular that \((1 - \beta)p^s\) is different from \(p^r\). However, an interesting case to consider is when agents becomes very patient. Then it is legitimate to guess that the allocation will converge to the Walrasian one, as it is, in some sense, equivalent to agents trading with each other very frequently. However, this is not the case: While it is true that agents can trade very fast and they may be able to replicate a meeting with the representative “market” agent, they are also bargaining a lot and this friction remains. Indeed, as \( \beta \) tends to one, the solution to the bargaining problem is characterized by \( \bar{a}_h, \bar{a}_t \to A \), so that asset sales converge to zero. Hence, we obtain

\[
\lim_{\beta \to 1} (1 - \beta)p^s = (1 - \theta)u'_h(A) + \theta u'_e(A).
\]

However in the limit \( q^r \) satisfies \( u'_h(A + q^r) = u'_e(A - q^r) \) and Assumption 1 guarantees that \( q^r > 0 \) is bounded away from zero. Since \( q^r > 0 \) and \( u_i(\cdot) \) is concave,

\[
q^r u'_h(A + q^r) < u_h(A + q^r) - u_h(A) < q^r u'_h(A)
\]

and in general \( p^r \neq (1 - \beta)p^s \). For illustration, we use the following utility function: \( u_h(a) = \frac{a^{1 - \sigma}}{1 - \sigma} \) and \( u_e(a) = \lambda u_h(a) \) where \( \lambda \in (0, 1) \). Then we obtain

\[
q^r = \frac{\lambda^{-\frac{1}{\sigma}} \bar{a}_t - \bar{a}_h}{1 + \lambda^{-\frac{1}{\sigma}}}
\]

so that

\[
\bar{a}_h + q^r = \lambda^{-\frac{1}{\sigma}} \frac{2A}{1 + \lambda^{-\frac{1}{\sigma}}}
\]
and

\[ \bar{a}_e - q^r = \frac{2A}{1 + \lambda^{1 - \sigma}} \]

As we have argued above, \( a_e \) and \( a_h \) tends to \( A \) whenever \( \beta \to 1 \). Therefore in this case,

\[ \lim_{\beta \to 1} (1 - \beta)p^s = (1 - \theta + \lambda \theta)A^{-\sigma}, \]

while

\[ \lim_{\beta \to 1} p^r = \frac{A^{-\sigma}}{1 - \sigma} \left( \frac{\lambda^{\frac{1}{\sigma}} + 1}{1 - \lambda^{\frac{1}{\sigma}}} \right)^{\sigma} \left( \frac{2(1 - \sigma) - (\lambda^{\frac{1}{\sigma}} + 1)^{1-\sigma}}{\lambda 2^{1-\sigma} - \lambda (\lambda^{\frac{1}{\sigma}} + 1)^{1-\sigma}} \right) . \]

With \( \lambda = 0.1 \) and \( \sigma = 2 \), we plot the ratio \( \lim_{\beta \to 1} (1 - \beta)p^s / \lim_{\beta \to 1} p^r \) as a function of \( \theta \in [0, 0.4] \).

As the figure shows, \( \lim_{\beta \to 1} (1 - \beta)p^s > \lim_{\beta \to 1} p^r \) for low values of \( \theta \) and the inequality is reversed otherwise.

Figure 1 shows how \( \bar{a}_h \) (red curve) and \( \bar{a}_e \) (blue curve) evolve as \( \pi \) varies from \( 1/2 \) to \( 1 \). The parameters chosen are \( \theta = 0.5, \lambda = 0.1, \sigma = 2, \beta = 0.9, \) and \( A = 50 \). Interestingly, the rate of divergence increases as types become more persistent. Hence, as \( \pi \) becomes large, we should expect some wide movements in prices and quantities.
This intuition is confirmed by Figure 2 that shows prices for repo \( p^r \) and asset sales \( p^s \).

Similarly, total repo volume \( q^r \) and total sales volume \((1 - \pi)q^s\) display very different patterns, as illustrated in Figure 3. At \( \pi = 0.9 \), the total volume of repo is approximately 20% of the outstanding securities, while total sales are only 1% of outstanding securities.
Interestingly, the coefficient of risk aversion $\sigma$ is the one with the most impact on asset volumes and values. With $A = 50$, we can match the observations on repos and sales of Treasury securities, with $\sigma = 0.5$ (close to risk neutrality) and quite high persistence, $\pi = 0.9$.\footnote{The average daily volume of Treasury repos is approximately twice the one for Treasury sales in the US according to ICAP, see \url{http://www.icap.com/investor-relations/monthly-volume-data.aspx}.}

8. Outside Option

An intuitive explanation for our results is that agents may prefer to use repos (to acquiring an asset), because they do not want to lock in a position that may be difficult to undo later at an agreeable price. When they engage in repos, agents are not locked into a position. To make this intuition more precise, we modify the environment slightly and assume that agents’ outside option is to access a Walrasian market from next period onward. Then the outside option for an agent $i$ holding $a$ units of the asset is $u_i(a) + \beta \tilde{W}_i(a)$ where $\tilde{W}_i(a) = \pi W_i(a) + (1 - \pi) W_{-i}(a)$ and $W(a)$ has been defined in Section 3. The possibility to trade on a Walrasian market would make the “lock-in” problem a little less severe, as agents could sell their securities on the walrasian market next period. Therefore we would expect the repo trade to decrease relative to the economy where agents do not have the option to unload their asset holdings on a Walrasian market. Still, agents are locked-in for one period and we would still expect repo to have a role. Indeed, let $\bar{q}^r$ be the equilibrium level of repo taking place in the economy where agents have the option to trade at Walrasian price in the next period, and let $q^r$ be the equilibrium level of repo when they do not have this option. Then, in the Appendix, we show

**Proposition 11.** With directed search and bargaining, there is an equilibrium where $q^r > \bar{q}^r > 0$.

The equilibrium with the option to trade on a Walrasian market displays the same feature the one in our original set-up. That is, whether agents switched types or not, they always repo $\bar{q}^r$. In addition, those agents who switched types trade $q^s = \bar{a}_h - \bar{a}_t$ and zero otherwise, where $\bar{a}_h$ and $\bar{a}_t$ are given by some equilibrium conditions. The important result is that, although agent’s outside option is the Walrasian price, agents will still use repos, but less so than if they did not have the option to trade at Walrasian price the following period. Therefore, the repo volume declines as we take the economy “closer” to its Walrasian benchmark.
9. Appendix

9.1. Proof of Proposition 4

To show Proposition 4 we need to know $V_h'$ and $V'_\ell$. Using the fact that the constraint on agents $\ell$ is always binding and the definition of value functions, we obtain

$$V_h'(a) = \pi \int [u_h'(a + q^s + q^r) (1 + \frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a}) - \frac{\partial d}{\partial a} + \beta V_h'(a + q^s)(1 + \frac{\partial q^s}{\partial a})] d\mu(\ell) + (1 - \pi)U'(a)$$

and

$$V'_\ell(a) = (1 - \pi) \int [u_h'(a + q^s + q^r) (1 + \frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a}) - \frac{\partial d}{\partial a} + \beta V'_\ell(a + q^s)(1 + \frac{\partial q^s}{\partial a})] d\mu(\ell) + \pi U'(a)$$

Since the participation constraint of the type 2 agents is always binding, we also know that changing the asset holding of agent of type 1 will not change the utility of type 2 agents, or

$$-u'_\ell(\ell - q^s - q^r) (\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a}) + \frac{\partial d}{\partial a} - \beta V'_\ell(\ell - q^s) \frac{\partial q^s}{\partial a} = 0$$

Therefore,

$$\frac{\partial d}{\partial a} = \beta V'_\ell(\ell - q^s) \frac{\partial q^s}{\partial a} + u'_\ell(\ell - q^s - q^r) (\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a})$$

replacing this expression in $V_h'$ and $V'_\ell$ and using the fact that in the pairwise core (4) holds, we obtain

$$V_h'(a) = \pi \int (\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a}) [u_h'(a + q^s + q^r) - u'_\ell(\ell - q^s - q^r)] + u'_h(a + q^s + q^r) + \beta V_h'(a + q^s) d\mu(\ell) + (1 - \pi)U'(a)$$

Using the first order conditions we obtain

$$\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a} = 0 \quad \text{if} \quad \xi_\ell = 0$$

$$\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a} = 0 \quad \text{if} \quad \xi_\ell > 0$$

$$\frac{\partial q^s}{\partial a} + \frac{\partial q^r}{\partial a} = -1 \quad \text{if} \quad \xi_h > 0$$

where the last two equalities follows from the fact that when $\xi_\ell > 0$ we have $q^s + q^r = a_\ell$.

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while when $\xi_h > 0$ we have $q^s + q^r = -a_h$ (and $u_h'(0) \leq u_h'(a_\ell + a)$). Therefore,

$$V_h'(a) = \pi \int_{\{a_\ell: \xi_h(a_\ell, a) > 0\}} [u_h'(0) - u_h'(a_\ell + a)] d\mu_\ell(a_\ell) + \pi \int u_h'(a + q^s + q^r) + \beta V_h'(a + q^s)d\mu_\ell(a_\ell)$$

and as the set of $a_\ell$ such that $\xi_h > 0$ has measure zero by Assumption 1, we obtain

$$V_h'(a) = \pi \int [u_h'(a + q^s + q^r) + \beta V_h'(a + q^s)] d\mu_\ell(a_\ell) + (1 - \pi)U'(a) \tag{15}$$

and using a similar argument, we obtain

$$V_\ell'(a) = (1 - \pi) \int [u_\ell'(a + q^s + q^r) + \beta V_\ell'(a + q^s)] d\mu_\ell(a_\ell) + \pi U'(a) \tag{16}$$

How does (15) and (16) change as $a$ is increasing? We have

$$\frac{\partial}{\partial a} V_h'(a + q^s) = V_h''(a + q^s)(1 + \frac{\partial q^s}{\partial a}) < 0$$

and

$$\frac{\partial}{\partial a} u_h'(a + q^s + q^r) = u_h''(a + q^s + q^r)(1 + \frac{\partial(q^s + q^r)}{\partial a}) < 0$$

Where the inequalities follow from the first order conditions, as we get

$$u_h''(a_h + q^s + q^r)(1 + \frac{\partial(q^s + q^r)}{\partial a_h}) = u_h''(a_\ell - q^s - q^r)(-\frac{\partial(q^s + q^r)}{\partial a_h})$$

so that

$$\frac{\partial(q^s + q^r)}{\partial a_h} = \frac{-u_h''}{u''_h + V_\ell''} \in (-1, 0)$$

and using (4) we also obtain (guessing that the value function is concave and differentiable)

$$\frac{\partial q^s}{\partial a_h} = \frac{-V_h''}{V_\ell'' + V_h''} \in (-1, 0)$$

Therefore, given $V_i'(a)$ is decreasing in $a$ (verifying the initial guess). Also, increasing $a_\ell$, we have

$$\frac{\partial}{\partial a_\ell} V_h'(a + q^s) = V_h''(a + q^s)\frac{\partial q^s}{\partial a_\ell} < 0$$

and

$$\frac{\partial}{\partial a_\ell} u_h'(a + q^s + q^r) = u_h''(a + q^s + q^r)\frac{\partial(q^s + q^r)}{\partial a_\ell} < 0$$
where the signs are given by the following argument: Using the properties of the core allocation we obtain that either \( \frac{\partial (q^s + q^r)}{\partial a_{\ell}} = 1 > 0 \) if \( \xi_{\ell} > 0 \) and otherwise
\[
u''(a_h + q^s + q^r) \frac{\partial (q^s + q^r)}{\partial a_{\ell}} = \nu''(a_{\ell} - q^s - q^r)(1 - \frac{\partial (q^s + q^r)}{\partial a_{\ell}})
\]
so that
\[
\frac{\partial (q^s + q^r)}{\partial a_{\ell}} = \frac{\nu''}{\nu'' + u''} \in (0, 1)
\]
and using (4) we also obtain (guessing that the value function is concave and differentiable)
\[
\frac{\partial q^s}{\partial a_{\ell}} = \frac{V''}{V'' + V''} \in (0, 1)
\]

9.2. \textit{Proof of} \( u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h) \)

First notice that the equilibrium payoff of an agent \( h \) holding \( \bar{a}_\ell \) is
\[
u_h(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h) + q^r(\bar{a}_\ell, \bar{a}_h)) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_\ell + q^s(\bar{a}_\ell, \bar{a}_h))
\]
\[
= \bar{u}_h - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)
\]
\[
= \bar{u}_h + \bar{u}_\ell + \beta V_{\ell}(\bar{a}_\ell) + \beta V_h(\bar{a}_h) - U(\bar{a}_h) \geq u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)
\]
where \( \bar{u}_h = u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) \) and \( \bar{u}_\ell = u_\ell(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell)) \) and where the inequality follows from the participation constraint. Adding \( u_h(\bar{a}_h) \) on both sides and rearranging terms, we must have
\[
\bar{u}_h + \bar{u}_\ell + \beta V_{\ell}(\bar{a}_\ell) + u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h) + u_h(\bar{a}_\ell) + u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)
\]
\[
(17)
\]
However, notice that
\[
\bar{u}_h + \bar{u}_\ell + \beta V_{\ell}(\bar{a}_\ell) < u_h(\bar{a}_h) + u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)
\]
\[
(18)
\]
To see this, observe that
\[
\bar{u}_h + \bar{u}_\ell \leq u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) + u_h(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell))
\]
\[
26
\]
(notice that we have changed the subscript from \( \ell \) to \( h \) in the last term, thus explaining the inequality sign), and concavity of the utility function guarantees that

\[
u_h(\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) + u_h(\bar{a}_\ell - q^r(\bar{a}_h, \bar{a}_\ell)) \leq u_h(\bar{a}_h) + u_h(\bar{a}_\ell)
\]

Hence, \( \bar{u}_h + \bar{u}_\ell \leq u_h(\bar{a}_h) + u_h(\bar{a}_\ell) \) while we also have \( V_h(a) \geq V_\ell(a) \) for all \( a \). Therefore (18) holds. Given (18), the only way that (17) can hold is that \( u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) \geq U(\bar{a}_h) \).

\[9.3. \text{Proof of Proposition 6}\]

From (15) as well as our direct matching mechanism, we obtain for any \((a_h, a_\ell)\)

\[
V'_h(a_h) = \pi[u'_h(a_h + q^r(a_h, a_\ell)) + \beta V'_h(a_h)] + (1 - \pi)U'(a_h)
\]

as an agent \( h \) who did not switch only enters into repos with an agent \( \ell \) who did not switch. Therefore,

\[
V'_h(a_h) = \frac{\pi}{1 - \beta \pi} u'_h(a_h + q^r(a_h, a_\ell)) + \frac{1 - \pi}{1 - \beta \pi} U'(a_h)
\]

Also, from (16) and the direct matching rule, we have for any \((a_h, a_\ell)\)

\[
V'_\ell(a_\ell) = (1 - \pi)[u'_h(a_\ell + q^s(a_\ell, a_h) + q^r(a_\ell, a_h))] + \beta V'_h(a_\ell + q^s(a_\ell, a_h))] + \pi U'(a_\ell)
\]

as a type \( \ell \) who switched is matched with a type \( h \) who just switched.

Using the pairwise core allocations \( q^s(a_\ell, a_h) = a_h - a_\ell \) and \( q^r(a_\ell, a_h) = q^r(a_h, a_\ell) \) we obtain

\[
V'_\ell(a_\ell) = (1 - \pi)[u'_h(a_\ell + q^r(a_\ell, a_h))] + \beta V'_h(a_h) + \pi U'(a_\ell)
\]

and using (20) and simplifying we obtain\(^{13}\)

\[
V'_\ell(a_\ell) = \frac{1 - \pi}{1 - \beta \pi} u'_h(a_\ell + q^r(a_h, a_\ell)) + \frac{\pi}{1 - \beta \pi} U'(a_\ell)
\]

Simplifying, we obtain \( V'_h(a_h) - V'_\ell(a_\ell) = 0 \) if and only if

\[
(2\pi - 1)V'_h(a_h) = \pi^2 U'(a_\ell) - (1 - \pi)^2 U'(a_h)
\]

\(^{13}\)or \( V'_\ell(a_\ell) = \frac{1 - \pi}{\pi}[V'_h(a_h) - (1 - \pi)U'(a_h)] + \pi U'(a_\ell) \).
Using equation (20) we simplify this expression to

\[ u_h'(a_h + q^*) = \frac{1}{(2\pi - 1)} \{(1 - \beta \pi)U'(a_\ell) - (1 - \beta(1 - \pi))(1 - \pi)U'(a_h)\} \quad (22) \]

as follows: Starting from

\[ (2\pi - 1)V_1'(a_h) = \pi^2U'(a_\ell) - (1 - \pi)^2U'(a_h) \]

use equation (20) to obtain

\[ \frac{\pi(2\pi - 1)}{1 - \beta \pi} u_h'(a_h + q^*) + \frac{(1 - \pi)(2\pi - 1)}{1 - \beta \pi} U'(a_h) = \pi^2U'(a_\ell) - (1 - \pi)^2U'(a_h) \]

so that

\[ \frac{\pi(2\pi - 1)}{1 - \beta \pi} u_h'(a_h + q^*) = \pi^2U'(a_\ell) - [(1 - \pi)^2 + \frac{(1 - \pi)(2\pi - 1)}{1 - \beta \pi}]U'(a_h) \]

And arranging,

\[
\begin{align*}
    u_h'(a_h + q^*) &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{1 - \beta \pi}{\pi(2\pi - 1)}[(1 - \pi)^2 + \frac{(1 - \pi)(2\pi - 1)}{1 - \beta \pi}]U'(a_h) \\
    &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{(1 - \beta \pi)(1 - \pi)^2}{\pi(2\pi - 1)} + \frac{(1 - \pi)}{\pi}U'(a_h) \\
    &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{(1 - \pi)(1 - \pi)(1 - \pi)}{\pi} + \frac{(2\pi - 1)}{\pi}U'(a_h) \\
    &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{(1 - \pi)(1 - \pi)(1 - \pi + (2\pi - 1))}{\pi}U'(a_h) \\
    &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{(1 - \pi)(1 - \pi)(1 - \pi + (2\pi - 1))}{\pi}U'(a_h) \\
    &= \frac{(1 - \beta \pi)\pi}{(2\pi - 1)} U'(a_\ell) - \frac{(1 - \pi)(1 - \pi)(1 - \pi + 1)}{\pi}U'(a_h) \\
    &= \frac{1}{(2\pi - 1)} \{(1 - \beta \pi)\pi U'(a_\ell) - (1 - \beta(1 - \pi))(1 - \pi)U'(a_h)\}
\end{align*}
\]
The conditions for the core equilibrium then become
\[
\begin{align*}
u_0' & (a_h + q^r) = u_2'(a_\ell - q^r) \\
u_1' & (a_h + q^r) = \frac{1}{(2\pi - 1)} \left\{ (1 - \beta \pi) U'(a_\ell) - (1 - \beta (1 - \pi))(1 - \pi) U'(a_h) \right\} \\
a_h + a_\ell &= A
\end{align*}
\]

9.4. Proof of Corollary 7

Notice that we can rewrite (13) as
\[
u_0' (\bar{a}_h + q^r(\bar{a}_h, \bar{a}_\ell)) = \alpha_\ell(\pi) U'(\bar{a}_\ell) - \alpha_h(\pi) U'(\bar{a}_h)
\]

where
\[
\alpha_\ell(\pi) = \alpha_h(\pi) = \frac{2\beta \pi (1 - \pi) - 1}{(2\pi - 1)^2} < 0
\]
Therefore, using \( \bar{a}_h + \bar{a}_\ell = A \) and the implicit function theorem, we have
\[
u''_h (1 + \frac{\partial q^r}{\partial a_h}) d\bar{a}_h = \alpha'(\pi)[U'(\bar{a}_\ell) - U'(\bar{a}_h)] d\pi - \alpha_\ell(\pi) U''_h (A - \bar{a}_h) + \alpha_h(\pi) U''_h (\bar{a}_h)] d\bar{a}_h \tag{23}
\]
Notice first that so that \( \frac{\partial q^r}{\partial a_h} = -1 \): Since \( u_h'(\bar{a}_h + q^r) = u_\ell'(A - \bar{a}_h - q^r) \) we have
\[
u''_h (1 + \frac{\partial q^r}{\partial a_h}) = -u_\ell''(1 + \frac{\partial q^r}{\partial a_h})
\]
since \( u_\ell'' < 0 \) while \( -u_h'' > 0 \) the only solution is that
\[
\frac{\partial q^r}{\partial a_h} = -1 \tag{24}
\]
Therefore using (23) we obtain
\[
\frac{d\bar{a}_h}{d\pi} = \frac{\alpha'(\pi)[U'(\bar{a}_\ell) - U'(\bar{a}_h)]}{\alpha_\ell(\pi) U''_h (A - \bar{a}_h) + \alpha_h(\pi) U''_h (\bar{a}_h)}
\]
Since both the denominator and the numerator are negative we have \( d\bar{a}_h/d\pi > 0 \).

Given \( \pi \) the volume of repos in this economy is given by \( q^r \) (since all agents use repo) while the volume of asset sales is given by \( (1 - \pi)q^s = (1 - \pi)(\bar{a}_h - \bar{a}_\ell) \). Clearly, the sales volume is hump shaped as when \( \pi = 1/2 \) we have \( \bar{a}_h = \bar{a}_\ell \) so that \( q^s = 0 \) while when \( \pi = 1 \,
\( q^* = 0 \) as well. However, \((1 - \pi)q^* > 0\) for all other values of \( \pi \). Since the problem is continuous, sales volume is hum-shaped. Also, (24) implies that the total volume of repo is declining in \( \pi \). Since there are no repo when \( \pi = 1 \), the volume of repo is declining to zero.

9.5. Proof of Proposition 8

We need to show that no 1 or 2 agent(s) wish(es) to form a coalition and be better off. It should be clear that no 1 agent wants to form a coalition (this option is already embedded in the bargaining problem).

Now, an agent \( \ell \) with \( \bar{a}_h \) could decide to form a coalition with an agent \( \ell \) with \( \bar{a}_\ell \) or an agent \( h \) with \( \bar{a}_h \). It is a property of the bargaining solution that an agent \( \ell \) will obtain a lower payoff being matched with an agent \( h \) with a higher amount of asset (he can extract less since the marginal utility of obtaining more of the asset is lower for this agent). Hence, an agent \( \ell \) with \( \bar{a}_h \) prefers to be matched with an agent \( h \) with \( \bar{a}_\ell \). Also, it is a property of the bargaining solution that, given he has to meet an agent with asset holdings \( a \), an \( \ell \) agents prefer to be matched with the agent with the highest marginal utility (so agent \( h \)).

Also, an agent \( h \) with \( \bar{a}_\ell \) could decide to form a coalition with an agent \( h \) with \( \bar{a}_h \) or an agent \( \ell \) with \( \bar{a}_\ell \). As above, however, it is a property of the bargaining solution that an agent \( h \) payoff matched with an agent \( \ell \) will get a higher utility whenever the agent \( \ell \) is holding more asset. Hence, the agent \( h \) will not want to be matched with an agent \( \ell \) holding \( \bar{a}_\ell \). Also, an agent \( h \) with \( \bar{a}_h \) prefers to be matched with the agent holding \( \bar{a}_\ell \) with the lowest marginal utility, i.e. with an \( \ell \) agent.

Hence there are no 2-agents coalition where both agents would do better than under the prescribed matching technology, which shows that it, together with the distribution over \( \{\bar{a}_h, \bar{a}_\ell\} \) is an equilibrium.

Proof of Proposition 9

The value functions are

\[
V_h(\bar{a}_h) = \pi [u_h(\bar{a}_h + q^*) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] + (1 - \pi)[u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)]
\]

\[
V_\ell(\bar{a}_\ell) = \pi [u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi)[u_h(\bar{a}_h + q^*) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)]
\]

\footnote{It is easy to show this with \( u_h(a) = \alpha u_\ell(a) \) with \( \alpha > 1 \): Compute the bargaining solution and show that \( \partial [u_\ell(a_\ell - q^* - q^*) + \beta V(a - q^*) + d] / \partial \alpha > 0 \).}
Adding both equations, we obtain

\[ V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) = \frac{u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*)}{1 - \beta} \]  

(25)

Also

\[ V_h(\bar{a}_\ell) = \pi [u_h(\bar{a}_h + q^*) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h)] + (1 - \pi) [u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] \]

\[ V_\ell(\bar{a}_h) = \pi [u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi) [u_h(\bar{a}_h + q^*) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h)] \]

and adding both equations, we obtain also

\[ V_h(\bar{a}_h) + V_\ell(\bar{a}_h) = V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) \]

(26)

From the bargaining first order condition, we obtain

\[ d(a_h, a_\ell) = (1 - \theta) [u_h(a_h + q^* + q^*) - u_h(a_h) + \beta V_h(a_h + q^*) - \beta V_h(a_h)] - \theta [u_\ell(a_\ell - q^* - q^*) - u_\ell(a_\ell) + \beta V_\ell(a_\ell - q^*) - \beta V_\ell(a_\ell)] \]

so that the transfer \( d(\bar{a}_h, \bar{a}_\ell) \) is

\[ d(\bar{a}_h, \bar{a}_\ell) = (1 - \theta) [u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h + q^*) - \beta V_h(\bar{a}_h)] - \theta [u_\ell(\bar{a}_\ell - q^* - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell - q^*) - \beta V_\ell(\bar{a}_\ell)] \]

\[ = (1 - \theta) [u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h)] - \theta [u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell)] \]

where we have used the fact that \( q^*(\bar{a}_h, \bar{a}_\ell) = 0 \). Therefore, we obtain

\[ u_h(\bar{a}_h + q^*) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) = \]

\[ u_h(\bar{a}_h + q^*) - (1 - \theta) [u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h)] + \theta [u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell)] + \beta V_h(\bar{a}_h) = \]

\[ u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) + \theta [u_h(\bar{a}_h + q^*) - u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)] \]

Also

\[ u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) = \]

\[ u_\ell(\bar{a}_\ell - q^*) + (1 - \theta) [u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h)] - \theta [u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell)] + \beta V_\ell(\bar{a}_\ell) = \]

\[ u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) + (1 - \theta) [u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)] \]
In a similar fashion, we obtain (using \( q^*(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell \))

\[
d(\bar{a}_\ell, \bar{a}_h) = (1 - \theta)[u_h(\bar{a}_h + q^r) - u_h(\bar{a}_\ell)] + \beta V_h(\bar{a}_h) - \beta V_h(\bar{a}_\ell) - \theta[u_\ell(\bar{a}_\ell - q^r) - u_\ell(\bar{a}_h)] + \beta V_\ell(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)
\]

It is easy to rewrite it as

\[
d(\bar{a}_\ell, \bar{a}_h) = d(\bar{a}_h, \bar{a}_\ell) + \bar{u} + \beta(1 - \theta)[V_h(\bar{a}_h) - V_h(\bar{a}_\ell)] + \beta[V_\ell(\bar{a}_h) - V_\ell(\bar{a}_\ell)]
\]

where

\[
\bar{u} = (1 - \theta)[u_h(\bar{a}_h) - u_h(\bar{a}_\ell)] + \theta[u_\ell(\bar{a}_\ell) - u_\ell(\bar{a}_h)]
\]

Therefore,

\[
u_\ell(\bar{a}_\ell - q^r) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell) = \]

\[
u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h) + (1 - \theta)[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) + \beta V_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h) - \beta V_h(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)] = \]

\[
u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h) + (1 - \theta)[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h)] = \]

where we have used (26), and similarly

\[
u_h(\bar{a}_h + q^r) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h) = \]

\[
u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) + \theta[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) + \beta V_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h) - \beta V_h(\bar{a}_\ell) - \beta V_\ell(\bar{a}_h)] = \]

\[
u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell) + \theta[u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h)] = \]

Hence, we obtain

\[
V_h(\bar{a}_h) = \pi[u_h(\bar{a}_h) + \beta V_h(\bar{a}_h)] + (1 - \pi)[u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h)] + \pi \theta \bar{S} + (1 - \pi)(1 - \theta)\tilde{S}
\]

\[
V_\ell(\bar{a}_\ell) = \pi[u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi)[u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)] + \pi(1 - \theta)\bar{S} + (1 - \pi)\theta \tilde{S}
\]

\[
V_h(\bar{a}_\ell) = \pi[u_h(\bar{a}_\ell) + \beta V_h(\bar{a}_\ell)] + (1 - \pi)[u_h(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell)] + (1 - \pi)(1 - \theta)S + \pi \theta \tilde{S}
\]

\[
V_\ell(\bar{a}_h) = \pi[u_\ell(\bar{a}_h) + \beta V_\ell(\bar{a}_h)] + (1 - \pi)[u_\ell(\bar{a}_h) + \beta V_h(\bar{a}_h)] + (1 - \pi)\theta S + \pi(1 - \theta)\tilde{S}
\]

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where

\[ S = u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell) \]
\[ \tilde{S} = u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_h) \]

Solving for \( V_h(\bar{a}_h) \) we obtain

\[ (1 - \beta)V_h(\bar{a}_h) = \frac{(1 - \pi)[u_\ell(\bar{a}_h) + (1 - \theta)\tilde{S}] + [\pi - (2\pi - 1)\beta][u_h(\bar{a}_h) + \theta S]}{1 - (2\pi - 1)\beta} \]

And taking the derivative, we have

\[ (1 - \beta)V'_h(\bar{a}_h) = \frac{u'_\ell(\bar{a}_h)(1 - \pi) + [\pi - (2\pi - 1)\beta]u'_h(\bar{a}_h) + (1 - \pi)(1 - \theta)\frac{\partial \tilde{S}}{\partial \bar{a}_h} + \theta \frac{\partial S}{\partial \bar{a}_h}[\pi - (2\pi - 1)\beta]}{1 - (2\pi - 1)\beta} \]

and using the first order condition for \( q^r \) we obtain

\[ (1 - \beta)(1 - (2\pi - 1)\beta)V'_h(\bar{a}_h) = u'_\ell(\bar{a}_h)(1 - \pi) + [\pi - (2\pi - 1)\beta]u'_h(\bar{a}_h) + (1 - \pi)(1 - \theta)[u'_h(\bar{a}_h + q^r) - u'_\ell(\bar{a}_h)] \]
\[ + \theta[u'_h(\bar{a}_h + q^r) - u'_h(\bar{a}_h)][\pi - (2\pi - 1)\beta] \]

so that after some simplifications,

\[ (1 - \beta)(1 - (2\pi - 1)\beta)V'_h(\bar{a}_h) = u'_\ell(\bar{a}_h)\theta(1 - \pi) + u'_h(\bar{a}_h)(1 - \theta)[\pi - (2\pi - 1)\beta] \]
\[ + u'_h(\bar{a}_h + q^r)[1 - \pi + (2\pi - 1)(1 - \beta)\theta] \]
Since (25) holds, and using the first order condition for \( q^r \) we obtain

\[
(1 - \beta)(1 - (2\pi - 1)\beta) V_\ell'(\bar{a}_\ell) = u'_\ell(\bar{a}_\ell - q^r)(1 - (2\pi - 1)\beta) - (1 - \beta)(1 - (2\pi - 1)\beta) \frac{\partial V_h(\bar{a}_h)}{\partial \bar{a}_\ell}
\]

\[
(1 - \beta)(1 - (2\pi - 1)\beta) V_\ell'(\bar{a}_\ell) = u'_\ell(\bar{a}_\ell - q^r)(1 - (2\pi - 1)\beta)
\]

\[-(1 - \pi)(1 - \theta) \frac{\partial S}{\partial \bar{a}_\ell} - \theta \frac{\partial S}{\partial \bar{a}_\ell} [\pi - (2\pi - 1)\beta]
\]

\[
(1 - \beta)(1 - (2\pi - 1)\beta) V_\ell'(\bar{a}_\ell) = u'_\ell(\bar{a}_\ell - q^r)(1 - (2\pi - 1)\beta)
\]

\[-u'_\ell(\bar{a}_\ell - q^r)[1 - \pi + (2\pi - 1)(1 - \beta)]
\]

\[+(1 - \pi)(1 - \theta)u'_h(\bar{a}_\ell) + \theta[\pi - (2\pi - 1)\beta]u'_\ell(\bar{a}_\ell)
\]

\[
(1 - \beta)(1 - (2\pi - 1)\beta) V_\ell'(\bar{a}_\ell) = u'_\ell(\bar{a}_\ell - q^r)[\pi - (2\pi - 1)(\beta + (1 - \beta)\theta)] + (1 - \pi)(1 - \theta)u'_h(\bar{a}_\ell)
\]

\[+\theta[\pi - (2\pi - 1)\beta]u'_\ell(\bar{a}_\ell)
\]

The first condition for \( q^s \) imposes that \( V_\ell'(\bar{a}_h) = V_\ell'(\bar{a}_\ell) \). Using the fact that \( u'_\ell(\bar{a}_\ell - q^r) = u'_h(\bar{a}_h + q^r) \) and simplifying, we obtain

\[
u'_h(\bar{a}_h + q^r) = \left[\frac{\pi - (2\pi - 1)\beta[\theta u'_\ell(\bar{a}_\ell) - (1 - \theta)u'_h(\bar{a}_h)] - (1 - \pi)[\theta u'_\ell(\bar{a}_h) - (1 - \theta)u'_h(\bar{a}_\ell)]}{(2\pi - 1)(1 - \beta)(2\theta - 1)}\right]
\]

Together with the first order condition on asset sales and the feasibility constraint, this completes the proof.

9.6. Proof of Proposition 11

We still assume that agents who did not switch are matched together, while those agents who just switched are matched with each other. With Nash bargaining, the allocation of an agent \( h \) with portfolio \( a_h \) matched with an agent \( \ell \) with portfolio \( a_\ell \) solves the following problem:

\[
\max_{q^*, q^r, d^1} [u_h(a_h + q^s + q^r) - d + \beta V_h(a_h + q^s) - u_h(a_h) - \beta W_h(a_h)]^\theta
\]

\[
	imes [u_\ell(a_\ell - q^s - q^r) + d + \beta V_\ell(a_\ell - q^s) - u_\ell(a_\ell) - \beta W_\ell(a_\ell)]^{1-\theta}
\]
with first order conditions

\[ V'_h(a_h + q^s) = V'_{\ell}(a_{\ell} - q^s) \]

\[ u'_h(a_h + q^s + q^r) = u'_{\ell}(a_{\ell} - q^s - q^r) \]

\[ d(h, \ell) = (1 - \theta)[u_h(a_h + q^s + q^r) + \beta V_h(a_h + q^s) - u_h(a_h) - \beta \tilde{W}_h(a_h)] \]

\[-\theta[u_{\ell}(a_{\ell} - q^s - q^r) + \beta V_{\ell}(a_{\ell} - q^s) - u_{\ell}(a_{\ell}) - \beta \tilde{W}_{\ell}(a_{\ell})] \]

We still assume that agents who did not switch types are matched together while those agents who just switched are matched together. We first solve for \( \tilde{W}_i(a) \). By definition, \( \tilde{W}_i(a) = \pi W_i(a) + (1 - \pi)W_j(a) \) with \( i \neq j \in \{h, \ell\} \) and where \( W_i \) denotes the value of participating in the Walrasian market as a type \( i \). From the problem of agents in the Walrasian market, it should be clear that \( W_i(a) = pa + W_i(0) \), where \( W_i(0) \) is given by

\[ \tilde{W}_i(0) = u_i(a_i^w) - pa_i^w + \beta E_{k_1}W_k(a_i^w) \]

\[ = u_i(a_i^w) - pa_i^w + \beta E_{k_1}[pa_i^w + W_k(0)] \]

\[ = u_i(a_i^w) - (1 - \beta)pa_i^w + \beta E_{k_1}W_k(0) \]

\[ = u_i(a_i^w) - p^r a_i^w + \beta E_{k_1}W_k(0) \]

where \( a_i^w \) is the solution to \( u'_i(a_i^w) = p^r \) and \( u'_h(a_h^w) = u'_{\ell}(a_{\ell}^w) \) with \( a_i^w + a_h^w = 2A \). Solving for \( W_i(0) \) we have

\[ W_h(0) = u_h(a_h^w) - p^r a_h^w + \beta \pi W_h(0) + \beta (1 - \pi)W_{\ell}(0) \]

\[ W_{\ell}(0) = u_{\ell}(a_{\ell}^w) - p^r a_{\ell}^w + \beta \pi W_{\ell}(0) + \beta (1 - \pi)W_h(0) \]

so that

\[ W_h(0) = \frac{(1 - \beta \pi)}{(1 - \beta \pi)^2 - \beta^2(1 - \pi)^2}[u_h(a_h^w) - p^r a_h^w]\]

\[ + \frac{\beta(1 - \pi)}{(1 - \beta \pi)^2 - \beta^2(1 - \pi)^2}[u_{\ell}(a_{\ell}^w) - p^r a_{\ell}^w]\]

after arranging we obtain

\[ (1 - \beta)W_h(0) = \frac{1 - \beta \pi}{1 + \beta - 2\beta \pi}[u_h(a_h^w) - p^r a_h^w]\]

\[ + \frac{\beta - \beta \pi}{1 + \beta - 2\beta \pi}[u_{\ell}(a_{\ell}^w) - p^r a_{\ell}^w]\]

so that

\[ (1 - \beta)W_h(0) = \alpha[u_h(a_h^w) - p^r a_h^w] + (1 - \alpha)[u_{\ell}(a_{\ell}^w) - p^r a_{\ell}^w]\]
where $\alpha \in [0, 1]$ is the obvious weight. Similarly,

$$(1 - \beta)W_t(0) = \alpha[u_t(a^w_t) - p^r a^w_t] + (1 - \alpha)[u_h(a^w_h) - p^r a^w_h]$$

Therefore,

$$(1 - \beta)\bar{W}_h(0) = \pi(1 - \beta)W_h(0) + (1 - \pi)(1 - \beta)W_t(0)$$

$$= \pi\alpha[u_h(a^w_h) - p^r a^w_h] + \pi(1 - \alpha)[u_t(a^w_t) - p^r a^w_t] + (1 - \pi)\alpha[u_h(a^w_h) - p^r a^w_h] + (1 - \pi)(1 - \alpha)[u_t(a^w_t) - p^r a^w_t]$$

$$= [\pi\alpha + (1 - \pi)(1 - \alpha)][u_h(a^w_h) - p^r a^w_h] + [\pi(1 - \alpha) + (1 - \pi)\alpha][u_t(a^w_t) - p^r a^w_t]$$

$$= [u(a^w_t) - p^r a^w_t + [\pi + \alpha - 2\pi\alpha][u_t(a^w_t) - p^r a^w_t - u_h(a^w_h) + p^r a^w_h]$$

and

$$(1 - \beta)\bar{W}_t(0) = (1 - \pi)(1 - \beta)W_h(0) + \pi(1 - \beta)W_t(0)$$

$$= u(a^w_t) - p^r a^w_t + [\pi + \alpha - 2\pi\alpha][u_h(a^w_h) - p^r a^w_h - u_t(a^w_t) + p^r a^w_t]$$

Notice that

$$\bar{W}_h(0) + \bar{W}_t(0) = \frac{u(a^w_h) - p^r a^w_h + u(a^w_t) - p^r a^w_t}{1 - \beta}$$

In this equilibrium we have (from the FOC of the bargaining problem)

$$u'_h(\bar{a}_h + q^r) = u'_t(\bar{a}_t - q^r)$$

so that

$$\bar{a}_h + q^r = a^w_h,$$

$$\bar{a}_t - q^r = a^w_t.$$
Adding both equations, we obtain

\[ V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) = \frac{u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*)}{1 - \beta} = \bar{W}_h(a_h^w) + \bar{W}_\ell(a_\ell^w) \] (27)

From the bargaining first order condition, we obtain

\[ d(a_h, a_\ell) = (1 - \theta)[u_h(a_h + q^* + q^r) - u_h(a_h) + \beta V_h(a_h + q^*) - \beta \bar{W}_h(a_h)] - \theta[u_\ell(a_\ell - q^* - q^r) - u_\ell(a_\ell) + \beta V_\ell(a_\ell - q^*) - \beta \bar{W}_\ell(a_\ell)] \]

so that the transfer \( d(\bar{a}_h, \bar{a}_\ell) \) is (using the fact that \( q^*(\bar{a}_h, \bar{a}_\ell) = 0 \),

\[ d(\bar{a}_h, \bar{a}_\ell) = (1 - \theta)[u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) - \beta \bar{W}_h(\bar{a}_h)] - \theta[u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_\ell)] = (1 - \theta)[u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) - \beta[p\bar{a}_h + \bar{W}_h(0)]] - \theta[u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) - \beta[p\bar{a}_\ell + \bar{W}_\ell(0)]] \]

Therefore, we obtain (using the relation between \( \bar{a}_i \) and \( a_i^w \) as well as equation (27)):

\[ u_h(\bar{a}_h + q^*) - d(\bar{a}_h, \bar{a}_\ell) + \beta V_h(\bar{a}_h) = u_h(\bar{a}_h + q^*) + \beta V_h(\bar{a}_h) - \]

\[ (1 - \theta)[u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) - \beta \bar{W}_h(\bar{a}_h)] + \theta[u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_\ell)] = u_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h) + \]

\[ \theta \left\{ u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) - u_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_\ell) - \beta V_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h) \right\} = u_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h) + \]

\[ \theta \left\{ u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) - u_h(\bar{a}_h) \right\} = \]
Also

\[ u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_h, \bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) = u_\ell(\bar{a}_\ell - q^*) + \beta V_\ell(\bar{a}_\ell) + \\
(1 - \theta)\left[u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) - \beta \bar{W}_h(\bar{a}_h)\right] - \\
\theta\left[u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_\ell)\right] = \\
u_\ell(\bar{a}_\ell) + \beta \bar{W}_\ell(\bar{a}_\ell) + \\
(1 - \theta)\left\{u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) - u_h(\bar{a}_h)\right\} \]

In a similar fashion, we obtain (using \(q^*(\bar{a}_\ell, \bar{a}_h) = \bar{a}_h - \bar{a}_\ell\))

\[ d(\bar{a}_\ell, \bar{a}_h) = (1 - \theta)\left[u_h(\bar{a}_h + q^*) - u_h(\bar{a}_h) + \beta V_h(\bar{a}_h) - \beta \bar{W}_h(\bar{a}_h)\right] - \\
-\theta\left[u_\ell(\bar{a}_\ell - q^*) - u_\ell(\bar{a}_\ell) + \beta V_\ell(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_\ell)\right] \]

Therefore,

\[ u_\ell(\bar{a}_\ell - q^*) + d(\bar{a}_\ell, \bar{a}_h) + \beta V_\ell(\bar{a}_\ell) = \\
u_\ell(\bar{a}_\ell) + \beta \bar{W}_\ell(\bar{a}_\ell) + (1 - \theta)\left[u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)\right] + \\
(1 - \theta)\beta [V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) - \bar{W}_h(\bar{a}_\ell) - \bar{W}_\ell(\bar{a}_h)] = \\
u_\ell(\bar{a}_\ell) + \beta \bar{W}_\ell(\bar{a}_\ell) + (1 - \theta)\left[u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)\right] + \\
(1 - \theta)\beta [V_h(\bar{a}_h) + V_\ell(\bar{a}_\ell) - p\bar{a}_\ell - \bar{W}_h(0) - p\bar{a}_h - \bar{W}_\ell(0)] = \\
u_\ell(\bar{a}_\ell) + \beta \bar{W}_\ell(\bar{a}_\ell) + (1 - \theta)\left\{u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell)\right\} \]

and similarly

\[ u_h(\bar{a}_h + q^*) - d(\bar{a}_\ell, \bar{a}_h) + \beta V_h(\bar{a}_h) = \\
u_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h) + \theta[u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) + \\
+\beta V_h(\bar{a}_h) + \beta V_\ell(\bar{a}_\ell) - u_h(\bar{a}_h) - u_\ell(\bar{a}_h) - \beta \bar{W}_h(\bar{a}_\ell) - \beta \bar{W}_\ell(\bar{a}_h)] = \\
u_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h) + \theta[u_h(\bar{a}_h + q^*) + u_\ell(\bar{a}_\ell - q^*) - u_h(\bar{a}_h) - u_\ell(\bar{a}_h)] \]

Using the above calculations, we obtain
\[ V_h(\bar{a}_h) = \pi [u_h(\bar{a}_h) + \beta \bar{W}_h(\bar{a}_h)] + (1 - \pi) [u_\ell(\bar{a}_h) + \beta \bar{W}_\ell(\bar{a}_h)] + \theta \pi S + (1 - \theta)(1 - \pi) \tilde{S} \]

\[ V_\ell(\bar{a}_\ell) = \pi [u_\ell(\bar{a}_\ell) + \beta \bar{W}_\ell(\bar{a}_\ell)] + (1 - \pi) [u_h(\bar{a}_\ell) + \beta \bar{W}_h(\bar{a}_\ell)] + \pi (1 - \theta) S + (1 - \pi) \theta \tilde{S} \]

where

\[ S = u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_h) - u_\ell(\bar{a}_\ell) \]

\[ \tilde{S} = u_h(\bar{a}_h + q^r) + u_\ell(\bar{a}_\ell - q^r) - u_h(\bar{a}_\ell) - u_\ell(\bar{a}_h) \]

And taking the derivative, we have

\[ V_h'(\bar{a}_h) = \pi [u'_h(\bar{a}_h) + \beta p] + (1 - \pi) [u'_\ell(\bar{a}_h) + \beta p] + (1 - \pi)(1 - \theta) \frac{\partial \tilde{S}}{\partial \bar{a}_h} + \theta \pi \frac{\partial S}{\partial \bar{a}_h} \]

and using the first order condition for \( q^r \) we obtain

\[ V_h'(\bar{a}_h) = \beta p + (1 - \pi) u'_\ell(\bar{a}_h) + \pi u'_h(\bar{a}_h) \]

\[ + (1 - \pi)(1 - \theta) [u'_h(\bar{a}_h + q^r) - u'_\ell(\bar{a}_h)] \]

\[ + \theta \pi [u'_h(\bar{a}_h + q^r) - u'_h(\bar{a}_h)] \]

Also, using the first order condition for \( q^r \) we obtain

\[ V_\ell'(\bar{a}_\ell) = \beta p + \pi u'_\ell(\bar{a}_\ell) + (1 - \pi) u'_h(\bar{a}_\ell) \]

\[ + \pi (1 - \theta) [u'_\ell(\bar{a}_\ell - q^r) - u'_h(\bar{a}_\ell)] + (1 - \pi) \theta [u'_\ell(\bar{a}_\ell - q^r) - u'_h(\bar{a}_\ell)] \]

The first condition for \( q^s \) imposes that \( V_h'(\bar{a}_h) = V_\ell'(\bar{a}_\ell) \). Using the fact that \( u'_\ell(\bar{a}_\ell - q^r) = u'_h(\bar{a}_h + q^r) \) and simplifying, we obtain

\[ (1 - \pi) \theta u'_\ell(\bar{a}_h) + \pi (1 - \theta) u'_h(\bar{a}_h) + [(1 - \pi)(1 - \theta) + \theta \pi] u'_h(\bar{a}_h + q^r) = \]

\[ \pi \theta u'_\ell(\bar{a}_\ell) + (1 - \pi)(1 - \theta) u'_h(\bar{a}_\ell) + [\pi (1 - \theta) + (1 - \pi) \theta] u'_h(\bar{a}_h + q^r) \]
or

$$(1 - 2\pi)(1 - 2\theta)u_h'(\bar{a}_h + q^*) = \pi \theta u_\ell'(\bar{a}_\ell) + (1 - \pi)(1 - \theta)u_h'(\bar{a}_h) - (1 - \pi)\theta u_\ell'(\bar{a}_h) - \pi(1 - \theta)u_h'(\bar{a}_h)$$

or

$$u_h'(\bar{a}_h + q^*) = \frac{\pi \theta u_\ell'(\bar{a}_\ell) + (1 - \pi)(1 - \theta)u_h'(\bar{a}_h) - (1 - \pi)\theta u_\ell'(\bar{a}_h) - \pi(1 - \theta)u_h'(\bar{a}_h)}{(1 - 2\pi)(1 - 2\theta)}$$

Therefore the equilibrium is given by

$$u_h'(\bar{a}_h + q^*) = u_\ell'(\bar{a}_\ell - q^*)$$

$$\bar{a}_h + \bar{a}_\ell = 2A$$

$$u_h'(\bar{a}_h + q^*) = \frac{\pi \theta u_\ell'(\bar{a}_\ell) + (1 - \pi)(1 - \theta)u_h'(\bar{a}_h) - (1 - \pi)\theta u_\ell'(\bar{a}_h) - \pi(1 - \theta)u_h'(\bar{a}_h)}{(1 - 2\pi)(1 - 2\theta)}$$

Notice that $\beta$ does not impact the equilibrium allocation.

Also, suppose that $q^* = 0$ is an equilibrium. Then $u_h'(\bar{a}_h) = u_\ell'(\bar{a}_\ell)$ and

$$u_h'(\bar{a}_h)(1 - 2\theta) = (1 - \theta)u_\ell'(\bar{a}_\ell) - \theta u_\ell'(\bar{a}_h)$$

or

$$\theta [u_h'(\bar{a}_h) - u_\ell'(\bar{a}_h)] = (1 - \theta) [u_h'(\bar{a}_\ell) - u_h'(\bar{a}_h)]$$

However, since $\bar{a}_\ell \leq \bar{a}_h$ the RHS is positive, while the LHS is negative by assumption. Therefore, $q^* = 0$ cannot be an equilibrium.

Now we show that the amount of repo is actually lower under this arrangement. The equilibrium allocations under the benchmark and Walrasian outside-option, can be summarized by

$$(1 - 2\pi)(1 - \beta)(1 - 2\theta)u_h'(c_h^*)$$

$= [\pi(1 - \beta) + (1 - \pi)\beta] [\theta u_\ell'(c_\ell^* + q^*) - (1 - \theta)u_h'(c_h^* - q^*)]$

$- (1 - \pi) [\theta u_\ell'(c_h^* - q^*) - (1 - \theta)u_h'(c_\ell^* + q^*)]$

$= \pi(1 - \beta) [\theta u_\ell'(c_\ell^* + q^*) - (1 - \theta)u_h'(c_h^* - q^*)]$

$- (1 - \pi)(1 - \beta) [\theta u_\ell'(c_h^* - q^*) - (1 - \theta)u_h'(c_\ell^* + q^*)]$
where \( u'_h(c^*_h) = u'_\ell(c^*_\ell) \) with \( c^*_h + c^*_\ell = 2A \) and \( q^{**} \) and \( \bar{q}^* \) are the repo amounts under the benchmark and Walrasian outside-option respectively. Define

\[
LH(q^*) = \theta u'_\ell(c^*_\ell + q^*) - (1 - \theta) u'_h(c^*_h - q^*)
\]

and

\[
RH(q^*) = \theta u'_\ell(c^*_h - q^*) - (1 - \theta) u'_h(c^*_\ell + q^*).
\]

Note that for \( q^* \leq (c^*_h - c^*_\ell) / 2 \), we have

\[
u'_\ell(c^*_h - q^*) < u'_\ell(c^*_\ell + q^*) < u'_\ell(c^*_\ell) < u'_h(c^*_h - q^*) < u'_h(c^*_\ell + q^*)
\]

therefore we have \( LH(q^*) > RH(q^*) \), which implies for all \( q^* \),

\[
[\pi(1 - \beta) + (1 - \pi)\beta] LH(q^*) - (1 - \pi)RH(q^*) > \pi(1 - \beta)LH(q^*) - (1 - \pi)(1 - \beta)RH(q^*)
\]

Finally concavity of \( u_h \) and \( u_\ell \) implies

\[
\frac{d}{dq^*} LH(q^*) < 0 < \frac{d}{dq^*} RH(q^*),
\]

hence both \([\pi(1 - \beta) + (1 - \pi)\beta] LH(q^*) - (1 - \pi)RH(q^*)\) and \(\pi(1 - \beta)LH(q^*) - (1 - \pi)(1 - \beta)RH(q^*)\) are both decreasing in \( q^* \). This means we have

\[
(1 - 2\pi)(1 - \beta)(1 - 2\theta) u'_h(c^*_h) = [\pi(1 - \beta) + (1 - \pi)\beta] [\theta u'_\ell(c^*_\ell + q^{**}) - (1 - \theta) u'_h(c^*_h - q^{**})] \\
- (1 - \pi) [\theta u'_\ell(c^*_h - q^{**}) - (1 - \theta) u'_h(c^*_\ell + q^{**})] \\
> \pi(1 - \beta) [\theta u'_\ell(c^*_\ell + q^{**}) - (1 - \theta) u'_h(c^*_h - q^{**})] \\
- (1 - \pi)(1 - \beta) [\theta u'_\ell(c^*_h - q^{**}) - (1 - \theta) u'_h(c^*_\ell + q^{**})]
\]

therefore \( q^{**} > \bar{q}^* \). Note that the change in \( \beta \) does not affect \( \bar{q}^* \), but as \( \beta \) approaches to zero, then \( q^{**} \to \bar{q}^* \).