Merger Policy with Merger Choice

Volker Nocke                  Michael D. Whinston
University of Mannheim        Northwestern University

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Abstract

We analyze the optimal policy of an antitrust authority towards horizontal mergers when merger proposals are endogenous and firms choose which of several mutually exclusive mergers to propose. In our model, the optimal policy of an antitrust authority that seeks to maximize expected consumer surplus imposes a tougher standard on “larger” mergers, i.e., those involving firms with a larger pre-merger market share, or equivalently, leading to a larger naively-computed post-merger Herfindahl index. The optimal policy is a response to a bias in firms’ proposal incentives: firms always propose a larger merger when it is better for consumers than a smaller one, but sometimes will propose the larger one even when it is worse for consumers.

1 Introduction

The evaluation of proposed horizontal mergers involves a basic trade-off: mergers may increase market power, but may also create efficiencies. Whether a given merger should be approved depends, as first emphasized by Williamson (1968), on a balancing of these two effects.

In most of the literature discussing horizontal merger evaluation, the assumption is that a merger should be approved if and only if it improves welfare, whether that be aggregate surplus or just consumer surplus, as is in practice the standard adopted by most antitrust authorities [see, e.g., Farrell and Shapiro (1990), McAfee and Williams (1992)]. Implicitly, the antitrust authority is viewed as facing a one-time merger and has no ability to commit ex ante to its approval policy. In practice, neither of these assumptions may be descriptive of reality since one merger proposal may be followed by others and commitment to an approval rule may be possible either through legislation or through the (long-lived) antitrust authority’s reputation.

This paper contributes to a small literature that formally derives optimal merger approval rules. This literature started with Besanko and Spulber (1993), who discussed the optimal rule for an antitrust authority who cannot directly observe efficiencies but who recognizes that firms know this information and decide whether to propose a merger based on this knowledge. Other recent papers in this literature
include Armstrong and Vickers (2010), Nocke and Whinston (2010), Ottaviani and Wickelgren (2009), and Neven and Roller (2005).

In this paper, we focus on a static setting (thus ignoring dynamic issues) in which one “pivotal” firm may merge with one of a number of other firms who have differing initial marginal costs. These mergers are mutually exclusive, and each may result in a different, randomly drawn post-merger marginal cost due to merger-related synergies. The merger that is proposed is the result of a bargaining process among the firms. The antitrust authority observes the characteristics of the merger that is proposed, but neither the feasibility nor the characteristics of any mergers that are not proposed. However, the antitrust authority can commit ex ante to its merger approval rule.

We focus on an antitrust authority who wishes to maximize expected consumer surplus. Our main result characterizes the form of the antitrust authority’s optimal policy, which we show should impose a tougher standard on mergers involving larger merger partners (in terms of their pre-merger market share). Specifically, the minimal acceptable improvement in consumer surplus is strictly positive for all but the smallest merger partner, and is larger the greater is the merger partner’s pre-merger share. Since in our model a greater pre-merger share for the merger partner is equivalent to a larger naively-computed post-merger Herfindahl index (computed assuming that the merged firm’s post-merger share is the sum of the merger partners’ pre-merger shares and that the shares of outsiders do not change), another way to say this is that mergers that result in a larger naively-computed post-merger Herfindahl index must generate larger improvements in consumer surplus to be approved. The form of this optimal policy is a response to a fundamental bias that we show exists in firms’ proposal incentives: whenever a larger merger would create at least as large a gain for consumers as a smaller one, the larger one is proposed if both would be approved. However, if both would be approved, the larger merger will sometimes be proposed even when it is worse for consumers. The optimal policy therefore rejects some consumer surplus-enhancing larger mergers to induce firms to propose instead better smaller ones.

The closest papers to ours are Lyons (2003) and Armstrong and Vickers (2010). Lyons is the first to identify the issue that arises when firms may choose which merger to propose and to note that committing to a policy may therefore be valuable. Motivated by the horizontal merger problem, Armstrong and Vickers (2010) provide an elegant characterization of the optimal policy for a principal facing an agent who may propose a project, when the principal cannot observe the characteristics of unproposed projects and the projects are ex ante identical in terms of their distributions of possible outcomes. Our paper differs from Armstrong and Vickers (2010) primarily in its focus on the optimal treatment of mergers that differ in this ex ante sense. Moreover, a key issue in our paper – the bargaining process among firms – is absent in Armstrong and Vickers, as they assume there is a single agent.1

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1Our paper also contributes to the theoretical literature on delegated agency without transfers, which was initiated by Holmstrom (1984). Recent contributions [in addition to Armstrong and Vickers (2010)] include Martimort and Semenov (2006), Alonso and Matouschek (2008), Armstrong and Vickers (2010), and Che, Dessein, and Kartik (2010). A key difference between Che, Dessein, and Kartik (2010) and our paper is that they assume that the principal (antitrust authority) can condition its policy only on the identity of the proposed project (merger) but not on its characteristics.
The paper is also related to Nocke and Whinston (2010). That paper establishes conditions under which the optimal dynamic policy for an antitrust authority who wants to maximize discounted expected consumer surplus is a completely myopic policy, in which a merger is approved if and only if it does not lower consumer surplus at the time it is proposed. A key assumption for that result is that potential mergers are “disjoint,” in the sense that the set of firms involved in different possible mergers do not overlap. The present paper explores, in a static setting, the implications of relaxing that disjointness assumption, focusing on the polar opposite case in which all potential mergers are mutually exclusive.

The paper proceeds as follows. We describe our model in Section 2. In Section 3 we derive some basic properties characterizing the structure of the antitrust authority’s merger policy design problem, most importantly demonstrating the bias in firms’ proposal incentives. We also note how this same structure can be applied to settings other than our baseline model, such as with efficient bargaining, alternative welfare standards, and differentiated price competition. In Section 4, we derive our main result: the antitrust authority optimally imposes a tougher standard, in terms of the minimum increase in consumer surplus required for approval, the “larger” is the proposed merger. In Section 4, we show that the optimal policy may not have a cutoff structure and provide a condition for verifying whether it does. Assuming it does, we examine some comparative statics. We conclude in Section 6.

2 The Model

We consider a homogeneous goods industry in which firms compete in quantities (Cournot competition). Let \( N = \{0, 1, 2, ..., N\} \) denote the (initial) set of firms. All firms have constant returns to scale; firm \( i \)’s marginal cost is denoted \( c_i \). Inverse demand is given by \( P(Q) \). We impose standard assumptions on demand:

**Assumption 1.** For all \( Q \) such that \( P(Q) > 0 \), we have:

(i) \( P'(Q) + qP''(Q) < 0 \) for all \( q \in [0, Q] \);

(ii) \( \lim_{Q \to \infty} P(Q) = 0 \).

Under these conditions there exists a unique Nash equilibrium in quantities. Moreover, this equilibrium is “stable” [each firm \( i \)’s best-response function \( b_i(Q_{-i}) = \arg \max_{q_i} [P(Q_{-i} + q_i) - c_i] \) satisfies \( b_i'(Q_{-i}) \in (-1, 0) \) whenever \( b_i(Q_{-i}) > 0 \), where \( Q_{-i} = \sum_{j \neq i} q_j \) so that comparative statics are “well behaved” (if a subset of firms jointly produce less [resp., more] because of a change in their incentives to produce output, then equilibrium industry output will decrease [resp., increase]). The vector of output levels in the pre-merger equilibrium is given by \( q^0 = (q_0^0, q_1^0, ..., q_N^0) \), where \( q_i^0 \) is firm \( i \)’s quantity. For simplicity, we assume that pre-merger marginal costs are such that all firms in \( N \) are active in the pre-merger equilibrium, i.e., \( q_i^0 > 0 \) for all \( i \). Hence, each firm \( i \)’s output \( (i = 0, 1, ..., N) \) satisfies the (post-merger costs).
first-order condition
\[ P(Q^0) + P'(Q^0)q_i^0 = c_i. \]  

Aggregate output, price, consumer surplus, and firm \( i \)’s profit in the pre-merger equilibrium are denoted \( Q^0 \equiv \sum_i q_i^0, P^0 \equiv P(Q^0), CS^0 \equiv \int_0^{Q^0} P(s)ds - P^0Q^0, \) and \( \pi_i^0 \equiv [P^0 - c_i]q_i^0 \), respectively. Firm \( i \)’s market share is \( s_i^0 \equiv q_i^0/Q^0 \).

We suppose that there is a set \( K \) of \( K \) potential mergers, each between firm 0 (the “acquirer”) and a single merger partner (a “target”) \( k \in K \subseteq N \). There is a random variable \( \phi_k \in \{0,1\} \) that determines whether the merger between firm 0 and firm \( k \) is feasible (\( \phi_k = 1 \)) or not (\( \phi_k = 0 \)). We let \( \theta_k \equiv \Pr(\phi_k = 1) > 0 \) denote the probability that the merger is feasible. A feasible merger is described by \( M_k = (k, \tau_k) \), where \( k \) is the identity of the target and \( \tau_k \) the (realized) post-merger marginal cost, which is drawn from distribution function \( G_k \) with support \([l, h_k]\) and no mass points. The random draws of \( \phi_k \) and \( \tau_k \) are independent across mergers. (We assume a common lower bound \( l \) primarily to simplify the statement of our main result, Proposition 1; we remark after that result about the effects of relaxing this restriction.) The realized set of feasible mergers is denoted \( \mathfrak{K} \equiv \{M_k : \phi_k = 1\} \).

If merger \( M_k \) is implemented, the vector of outputs in the resulting post-merger equilibrium is denoted \( q(M_k) \equiv (q_0(M_k), ..., q_N(M_k)) \), where \( q_k(M_k) \) is the output of the merged firm, aggregate output is \( Q(M_k) \equiv \sum_i q_i(M_k) \), and firm \( i \)’s market share is \( s_i(M_k) \equiv q_i(M_k)/Q(M_k) \). We assume that all nonmerging firms remain active after any merger, so individual outputs satisfy the first-order condition
\[ P(Q(M_k)) + P'(Q(M_k))q_i(M_k) = c_i \]  

for the nonmerging firms \( i \neq 0, k \) and
\[ P(Q(M_k)) + P'(Q(M_k))q_k(M_k) = \tau_k \]  

for the merged firm. The post-merger profit of nonmerging firm \( i \) is given by \( \pi_i(M_k) \equiv [P(Q(M_k)) - c_i]q_i(M_k) \), and the merged firm’s profit by \( \pi_k(M_k) \equiv [P(Q(M_k)) - \tau_k]q_k(M_k) \). The induced change in consumer surplus is
\[ \Delta CS(M_k) \equiv \left\{ \int_0^{Q(M_k)} P(s)ds - P(Q(M_k))Q(M_k) \right\} - CS^0. \]

We will say that a merger \( M_k \) is \textit{CS-neutral} if \( \Delta CS(M_k) = 0 \), CS-increasing if \( \Delta CS(M_k) > 0 \), and CS-decreasing if \( \Delta CS(M_k) < 0 \). A merger is CS-nondecreasing (resp., nonincreasing) if it is not CS-decreasing (resp., CS-increasing). If no merger is implemented, the status quo (or “null merger”) obtains, which we denote by \( M^0 \), resulting in outcome \( q(M^0) \equiv q^0, s_i(M^0) \equiv q_i^0/Q^0, \) and \( \Delta CS(M^0) = 0 \).

We assume that if merger \( M_k, k \in \mathfrak{K} \), is proposed, the antitrust authority can observe all aspects of that merger and knows as well the pre-merger cost levels of all firms [which can be inferred using (1) from knowledge of the demand function and observation of pre-merger sales]. What it does \textit{not} observe are the characteristics of any feasible mergers that are not proposed. We also assume that the antitrust
authority can commit ex ante to its policy. As such, the antitrust authority commits to a merger-
specific approval policy by specifying an approval (or “acceptance”) set \( A \equiv \{ M_k : c_k \in A_k \} \cup M^0 \), where \( A_k \subseteq [l, h_k] \) for \( k \in K \) are the post-merger marginal cost levels that would lead to approval of a merger with target \( k \). Because of our assumption of full support and no mass points, we can without loss of generality restrict attention to the case in which each \( A_k \) is a (finite or infinite) union of closed intervals possessing nonempty interiors, i.e., \( A_k = \cup_{r=1}^{R_k} [l_k^r, h_k^r] \), where \( l \leq l_k^r < h_k^r \leq h_k \) (\( R \) can be infinite). Note that the status quo “null merger” \( M^0 \) is always “approved.”

Some remarks are in order concerning the policies that we consider: First, we confine attention to deterministic policies. One justification is that it may be hard for the antitrust authority to commit to a random rule. Second, we do not pursue a mechanism design approach. Motivated by the constraints that antitrust authorities face in the real world, we assume that the antitrust authority cannot obtain verifiable information about mergers that are not proposed. Moreover, we assume that only one of the mutually exclusive mergers can be proposed to, and evaluated by, the antitrust authority.

Given a realized set of feasible mergers \( \mathfrak{F} \) and the antitrust authority’s approval set \( A \), the feasible mergers \( M_k \) that would be approved if proposed are given by the set \( \mathfrak{F} \cap A \). A bargaining process among the firms determines which feasible merger, if any, is actually proposed. Note that this bargaining problem involves externalities as firms’ payoffs depend on the identity of the target. We suppose that the bargaining process takes the form of an “offer game,” as in Segal (1999), where the acquirer (firm 0) – Segal’s principal – makes public take-it-or-leave-it offers. (However, see the end of Section 3 for a discussion of other bargaining processes and models of competition.)

In Segal (1999), the principal’s offers consist of a profile of “trades” \( x = (x_1, ..., x_K) \), with \( x_k \) the trade with agent \( k \). Here, \( x_k \in \{0, 1\} \), where \( x_k = 1 \) if the acquirer proposes a merger with firm \( k \). Hence, in our model Segal’s offer game simply amounts to firm 0 being able to make a take-it-or-leave-it offer of an acquisition price \( t_k \) to a single firm \( k \) of its choosing, where \( k \) is such that \( M_k \in (\mathfrak{F} \cap A) \). (Firm 0 can also choose to make no offer.) If the offer is accepted by firm \( k \), then merger \( M_k \) is proposed to the antitrust authority, who will approve it since \( M_k \in (\mathfrak{F} \cap A) \), and firm 0 acquires the target in return for the payment \( t_k \). If the offer is rejected, or if no offer is made, then no merger is proposed and no payments are made.

For \( k \in K \), let
\[ \Delta \Pi(M_k) \equiv \pi_k(M_k) - [\pi_0^0 + \pi_k^0] \]
denote the change in the bilateral (i.e., joint) profit of the merging parties, firms 0 and \( k \), induced by merger \( M_k \in (\mathfrak{F} \cap A) \). In what follows, it will also be convenient to define \( \Delta \Pi(M^0) \equiv 0 \), as no bilateral profit gain occurs when no merger happens. By choosing the payment \( t_k \) that makes firm \( k \) just indifferent between accepting and not, firm 0 can extract the entire bilateral profit gain \( \Delta \Pi(M_k) \).

\(^2\)The key restriction here is that the firms are always free to remain at the status quo. We include \( M^0 \) as an element of \( A \) to remind the reader of this assumption. Formally, it is equivalent to assume either that the null merger \( M^0 \) is always approved when proposed or that the status quo obtains if no merger \( M_k \) is proposed.
Given the realized set of feasible and acceptable mergers, \( \mathcal{F} \cap \mathcal{A} \), the merger outcome in the equilibrium of the offer game is therefore given by \( M^* (\mathcal{F}, \mathcal{A}) \), where

\[
M^* (\mathcal{F}, \mathcal{A}) = \begin{cases} 
\arg \max_{M_k \in (\mathcal{F} \cap \mathcal{A})} \Delta \Pi(M_k) & \text{if } \max_{M_k \in (\mathcal{F} \cap \mathcal{A})} \Delta \Pi(M_k) > 0 \\
M^0 & \text{otherwise.}
\end{cases}
\]

That is, the bargaining outcome is the feasible and allowable merger \( M_k \) that maximizes the induced change in the bilateral profit of firms 0 and \( k \), provided that change is positive; otherwise, the status quo \( M^0 \) remains intact.

In line with legal standards in the U.S., the EU, and many other jurisdictions, we assume that the antitrust authority acts in the consumers’ interests. That is, the antitrust authority selects the approval set \( \mathcal{A} \) that maximizes expected consumer surplus given that the bargaining outcome is \( M^* (\cdot) \):

\[
\max_{\mathcal{A}} E_{\mathcal{F}} [\Delta CS (M^* (\mathcal{F}, \mathcal{A}))],
\]

where the expectation is taken with respect to the set of feasible mergers, \( \mathcal{F} \). (See the end of Section 3 for a discussion of alternative welfare standards.) To make the antitrust authority’s problem interesting, and avoid certain degenerate cases, we will henceforth assume the following:

**Assumption 2.** For all \( k \in K \), the support of the post-merger cost distribution includes both CS-increasing and CS-nonincreasing mergers: i.e., \( \Delta CS(k, h_k) \leq 0 < \Delta CS(k, l) \).

We are interested in studying how the optimal approval set depends on the pre-merger characteristics of the alternative mergers. For this reason, we assume that the potential targets differ in their pre-merger marginal costs. To this end, we let \( K \equiv \{1, ..., K\} \) and label firms 1 through \( K \) in decreasing order of their pre-merger marginal costs: \( c_1 > c_2 > ... > c_K \). Thus, in the pre-merger equilibrium, firm \( k \in K \) produces more than firm \( j \in K \), and has a larger market share, if \( k > j \). We will say that merger \( M_k \) is larger than merger \( M_j \) if \( k > j \), as the combined pre-merger market share of firms 0 and \( k \) is larger than that of firms 0 and \( j \). Note also that the change in the naively-computed Herfindahl index (calculated using pre-merger shares) from a merger between firms 0 and \( k \) is \( 2s_0^2 s_k^2 \). Thus, a larger merger also causes a larger change in this naively-computed index.

### 3 Structure of the Merger Policy Decision

As firms produce a homogeneous good, a merger \( M_k \) raises consumer surplus if and only if it increases aggregate output \( Q \). The following lemma summarizes some useful properties of a CS-neutral merger \( M_k \), i.e., a merger that leaves consumer surplus unchanged [\( \Delta CS(M_k) = 0 \)]:

**Lemma 1.** Suppose merger \( M_k \) is CS-neutral. Then

\[
\text{Specifically, the change in the naively-computed Herfindahl index induced by merger } M_k \text{ is } \Delta H^{\text{naive}}(M_k) = \left( \sum_{i \neq 0, k} (s_i^2) + (s_0^2 + s_k^2)^2 \right) - \sum_{i = 0}^K (s_i^2)^2 = 2s_0^2 s_k^2.
\]
(i) the merger causes no changes in the output of any nonmerging firm \( i \notin \{0, k\} \) nor in the joint output of the merging firms 0 and \( k \);

(ii) the merged firm’s margin at the pre- and post-merger price \( P(Q^o) \) equals the sum of the merging firms’ pre-merger margins:

\[
P(Q^o) - \tau_k = [P(Q^o) - c_0] + [P(Q^o) - c_k];
\]

(iii) the merger is profitable for the merging firms: \( \Delta \Pi(M_k) > 0 \);

(iv) the merger increases aggregate profit:

\[
\sum_{i \in \mathcal{N} \setminus \{0\}} \pi_i(M_k) > \sum_{i \in \mathcal{N}} \pi_i^o.
\]

Proof. Part (i) follows from stability of equilibrium; part (ii) from the merged firm’s first-order condition for profit maximization (3) and from summing the merger partners’ pre-merger first-order conditions (1). Part (iii) is an implication of parts (i) and (ii): part (ii) implies that the margin earned on each sale is larger for the merged firm than it was pre-merger for either merger partner. Since, by (i), the total output of the merging firms does not change, their joint profit increases with the merger. As for part (iv), note that the merger raises the bilateral (i.e., joint) profit of the merging firms 0 and \( k \) by part (iii) and it leaves the profit of any nonmerging firm unchanged (as neither price nor their output changes).

Rewriting equation (4), merger \( M_k \) is CS-neutral if the post-merger marginal cost \( \tau_k \) satisfies

\[
\tau_k = \hat{\tau}_k(Q^o) \equiv c_k - [P(Q^o) - c_0].
\]

As the following standard lemma (proof omitted) shows, reducing the merged firm’s marginal cost \( \tau_k \) not only increases consumer surplus but also the profit of the merged firm:

**Lemma 2.** Conditional on merger \( M_k \) being implemented, a reduction in the post-merger marginal cost \( \tau_k \) causes aggregate output, consumer surplus, and the merged firm’s profit to increase.

Thus, conditional on merger \( M_k \) being implemented, both \( \Delta CS(M_k) \) and \( \Delta \Pi(M_k) \) – the changes in consumer surplus and bilateral profit of the merging firms – increase when the post-merger marginal cost declines. Combined with (5), this also implies that merger \( M_k \) is CS-increasing if \( \tau_k < \hat{\tau}_k(Q^o) \) and CS-decreasing if \( \tau_k > \hat{\tau}_k(Q^o) \).

The following lemma gives a key result that indicates that there is a systematic bias in the proposal incentives of firms in favor of larger mergers, relative to the interests of consumers. This bias arises in our model because an initially lower-cost firm, which produces more output, benefits more from any given cost reduction generated by a merger, and any two mergers that generate the same change in consumer surplus involve the same size cost reduction.

**Lemma 3.** Suppose two mergers, \( M_j \) and \( M_k \) with \( k > j \), induce the same non-negative change in consumer surplus, \( \Delta CS(M_j) = \Delta CS(M_k) \geq 0 \). Then the larger merger \( M_k \) induces a greater increase in the bilateral profit of the merger partners: i.e., \( \Delta \Pi(M_k) > \Delta \Pi(M_j) > 0 \).
Proof. We observe first that the result is true if mergers \( M_j \) and \( M_k \) are CS-neutral; i.e., if \( \Delta CS(M_j) = \Delta CS(M_k) = 0 \). In this case, parts (i) and (ii) of Lemma 1 imply that

\[
\Delta \Pi(M_k) - \Delta \Pi(M_j) = \left[ (P^o - c_k)q_k^o + (P^o - c_0)q_k^0 \right] - \left[ (P^o - c_j)q_j^o + (P^o - c_0)q_j^0 \right] > 0,
\]

where the inequality follows because \( (P^o - c_k) > (P^o - c_j) \) and \( q_k^o > q_j^o \).

We next show that as the post-merger aggregate output increases above \( Q^* \), \( \Delta \Pi(M_k) - \Delta \Pi(M_j) \) increases. To see this, define \( \bar{c}_i(Q) \) to be the post-merger cost level in merger \( i = j, k \) that results in output \( Q \). From the first-order conditions (2) and (3), this cost level satisfies

\[
NP(Q) + P'(Q)Q = \bar{c}_i(Q) + \sum_{r \in N \setminus \{0, i\}} c_r.
\]

Thus, \( c_k < c_j \) implies that \( \bar{c}_k(Q) < \bar{c}_j(Q) \) for all \( Q \). Also define \( q_i(Q) \) and \( Q_{-i}(Q) \) to be the output of the merged firm \( i = j, k \) and the aggregate output of all of its rivals in the associated equilibrium. The first-order conditions (2) imply that

\[
(N - 1)P(Q) + P'(Q)Q_{-i}(Q) = \sum_{r \in N \setminus \{0, i\}} c_r.
\]

Using the envelope theorem, the derivative of \( \Delta \Pi(M_i) \) for \( i = j, k \) with respect to a differential change in the post-merger aggregate output \( Q \) is

\[
\frac{d\Delta \Pi(M_i)}{dQ} = q_i(Q)[P'(Q)Q_{-i}(Q) - \bar{c}_i(Q)] = -q_i(Q)[2NP'(Q) + P''(Q)(2Q - q_i(Q)),
\]

where the second equality follows by substituting

\[
Q'_{-i}(Q) = -\left( \frac{1}{P'(Q)} \right) [(N - 1)P'(Q) + P''(Q)Q_{-i}(Q)]
\]

\[
\bar{c}_i'(Q) = [(N + 1)P'(Q) + P''(Q)Q],
\]

derived from (7) and (8). Holding \( Q \) fixed, expression (9) is larger the greater is the merged firm’s output \( q_i(Q) \):

\[
\frac{\partial}{\partial q} \left\{ -q[2NP'(Q) + P''(Q)(2Q - q)] \right\} = -2[NP'(Q) + P''(Q)(Q - q)] > 0,
\]

where the inequality follows from Assumption 1. Since \( \bar{c}_k(Q) < \bar{c}_j(Q) \) for all \( Q \), the first-order condition (3) implies that \( q_k(Q) > q_j(Q) \) for all \( Q \). Hence, (9) implies that \( d[\Delta \Pi(M_k) - \Delta \Pi(M_j)]/dQ > 0 \), which yields the result.

Lemmas 1 to 3 imply that the possible mergers can be represented as shown in Figure 1(a) (where there are four possible mergers; i.e., \( K = 4 \)). In the figure, the change in the merging firms’ bilateral profit, \( \Delta \Pi \), is measured on the horizontal axis and the change in consumer surplus, \( \Delta CS \), is measured on the vertical axis. The CS-increasing mergers therefore are those lying above the horizontal
Figure 1: In panel (a), each curve labeled $M_j$ depicts the relationship between the change in consumer surplus and the change in bilateral profit for a merger between firms 0 and $j$, with each point on the curve corresponding to a different realization of merger $M_j$’s post-merger marginal cost. Panel (b) depicts in heavy trace a possible merger approval policy $A$.

axis. The bilateral profit and consumer surplus changes induced by a merger between firms 0 and $k$, $(\Delta \Pi(M_k), \Delta CS(M_k))$, fall somewhere on the curve labeled “$M_k$.” (The figure shows only the parts of these curves for which the bilateral profit change $\Delta \Pi$ is nonnegative.) Since by Lemma 1 a CS-neutral merger is profitable for the merger partners, each curve crosses the horizontal axis to the right of the vertical axis. By Lemma 2, the curve for each merger $M_k$ is upward sloping. By Lemma 3, on and above the horizontal axis the curves for larger mergers lie everywhere to the right of those for smaller mergers. [In Figure 1(a) the curves remain ordered below the horizontal axis, but this need not be the case.]

Note that whenever a CS-nondecreasing merger has a greater bilateral surplus than a larger merger, it also must be better for consumers: i.e., if any two CS-nondecreasing mergers $M_j$ and $M_k$ with $k > j$ have $\Delta \Pi(M_k) \leq \Delta \Pi(M_j)$, then $\Delta CS(M_k) < \Delta CS(M_j)$ (this is proven formally as Corollary 1 in the Appendix). However, if instead the larger merger has a greater bilateral surplus [i.e., if $\Delta \Pi(M_k) > \Delta \Pi(M_j)$], consumers may be better off with the smaller merger.

Figure 1(b) shows a possible merger approval policy $A$: the mergers $M_k$ that would be approved
are contained in the sections of the curves with heavy trace (A also includes the origin, representing the status quo “null merger” $M^o$). Given this policy, for any given realization of feasible mergers $\mathfrak{F}$ such that some CS-nondecreasing merger is feasible, the proposed merger $M^* (\mathfrak{F}, A)$ is the one that lies furthest to the right (having the largest increase in bilateral profit).

The characterization of optimal merger policy we present in Sections 3 and 4 depends only on the structure of the antitrust authority’s policy design problem shown in Figure 1. As such, our characterization applies to many settings in addition to the one captured by our baseline model. (Readers interested only in our baseline model can skip ahead to Section 4.) In particular, suppose that each merger $M_k = (k, r_k)$ is summarized by the identity of the acquirer and a “characteristic” $r_k \in \mathbb{R}$, and results in a change in welfare $\Delta W(M_k)$ according to the antitrust authority’s objective.\footnote{We suppose as well that $r_k$ is drawn from a distribution with full support, that $\Delta W(M_k)$ is continuous in $r_k$, and that $\Delta W(M_k)$ can be either positive or negative with positive probability.} The status quo has $\Delta W(M^o) = 0$. As for bargaining, the offer game considered above is one example of what might be called a scoring-rule bargaining process. In a scoring-rule bargaining process, each merger $M_k$ has a score $S(M_k)$ that is continuous in $r_k$, and the merger that is proposed is the one with the highest score provided that is positive; otherwise the status quo $M^o$ (for which $S(M^o) = 0$) obtains; that is,

$$M^* (\mathfrak{F}, A) \equiv \begin{cases} \arg \max_{M_k \in (\mathfrak{F} \cap A)} S(M_k) & \text{ if } \max_{M_k \in (\mathfrak{F} \cap A)} S(M_k) > 0 \\ M^o & \text{ otherwise.} \end{cases}$$

(In the offer game, the score $S$ is the bilateral surplus $\Delta \Pi$.) A model with a scoring-rule bargaining process leads to the same structure for the antitrust authority’s problem as in our model (and a figure like Figure 1, but with $\Delta W$ on the vertical axis and $S$ on the horizontal axis) provided the following three properties hold:

- **Monotonicity:** $S(M_k)$ and $W(M_k)$ are decreasing in $r_k$ at all $r_k$ such that $\Delta W(M_k) \geq 0$.

- **Willingness to Propose:** $S(M_k) > 0$ if $\Delta W(M_k) = 0$.

- **Ordered Bias:** For any $k > j$, if $\Delta W(M_j) = \Delta W(M_k) \geq 0$ then $S(M_k) > S(M_j)$.

Monotonicity implies that a welfare-enhancing merger becomes more likely to be proposed if it becomes more attractive to the antitrust authority (due to a decrease in $r_k$). Willingness to Propose says that a welfare-neutral merger will be proposed if it is the only feasible merger. Combined with Monotonicity, it implies that the antitrust authority could achieve the first best if there were at most one feasible merger ($K = 1$) since any merger it would want to approve would be proposed in that case [i.e., $\Delta W(M_k) \geq 0$ implies $S(M_k) > 0$]. Ordered Bias says that there is an ordering of mergers such that the larger merger will be proposed when two mergers are equally attractive to the antitrust authority. It implies that the first best cannot be achieved if $K > 1$.\footnote{We suppose as well that $r_k$ is drawn from a distribution with full support, that $\Delta W(M_k)$ is continuous in $r_k$, and that $\Delta W(M_k)$ can be either positive or negative with positive probability.}
Our characterization results in Sections 3 and 4 can be applied whenever these three conditions are satisfied. For example, the following settings all satisfy these conditions:\(^5\)

**Efficient Bargaining.** Suppose that bargaining is instead efficient, selecting the merger that maximizes industry profit.\(^6\) Then the scoring rule \(S(\cdot)\) is simply the change in aggregate industry profit, which we denote by \(\Delta \Pi_i(M_k)\). (The welfare criterion \(\Delta W\) is still \(\Delta CS\).) There are several bargaining processes that would lead to aggregate profit maximization:

1. “Coasian bargaining” among all firms under complete information.

2. A “menu auction” in which each firm \(i \neq 0\) submits a nonnegative bid \(b_i(M_k) \geq 0\) to firm 0 for each merger \(M_k \in (\mathcal{F} \cap \mathcal{A})\) and the status quo \(M^0\), and firm 0 then selects the merger outcome that maximizes its profit, inclusive of these bids. [Firm 0’s profit from selecting the null merger \(M^0\) is \(\pi^0_0\).] Bernheim and Whinston (1996) show that there is an efficient equilibrium which, in this setting, implements the merger that maximizes aggregate profit.

3. The target (firm 0) committing to a sales mechanism. Jehiel, Moldovanu, and Stacchetti (1996) show that an optimal mechanism has the following structure in our setting: Firm 0 proposes to implement the aggregate profit-maximizing merger \(M^* (\mathcal{F}, \mathcal{A})\) and requires the payment \(\pi_i(M^* (\mathcal{F}, \mathcal{A})) - \pi_i(M)\) from each firm \(i \neq 0\), where \(M\) is the merger in set \((\mathcal{F} \cap \mathcal{A}) \setminus M_i\) that minimizes firm \(i\)’s profit. If a firm \(i\) does not accept participation in the mechanism when all other firms do, then the principal commits to proposing merger \(M\) to the antitrust authority.\(^7\) Given this mechanism, there is an equilibrium in which all firms participate in the mechanism, and merger \(M^* (\mathcal{F}, \mathcal{A})\) is proposed.\(^8\)

4. When \(K = N = 2\), a first-price sealed-bid auction for firm 0 in which each firm \(k \geq 1\) submits a non-negative bid. Whenever \(\mathcal{A} \subseteq \{M_k : \Delta W(M_k) \geq 0\}\) (as will always be the case), if two firms have feasible and allowable mergers, the unique undominated equilibrium of this auction has both

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\(^5\)In the Online Appendix we also identify situations in which these conditions hold for a collection of mutually exclusive mergers in which no single firm is part of all possible mergers. In addition, we discuss the case in which mergers may also result in reductions in fixed costs.

\(^6\)The case of efficient bargaining is closest to the model of Armstrong and Vickers (2010) since the industry then acts like a single agent in its proposal behavior. However, even in this single agent case our model differs from Armstrong and Vickers in the fact that a proposed merger may be drawn from one of \(K\) different identifiable distributions, whose distinct treatment is our central focus. A model similar to Armstrong and Vickers would emerge if instead a fixed number of merger “ideas” were drawn iid from a distribution over all \(K\) curves (with a given firm able to receive more than one idea). However, such a model would feature an unattractive negative correlation in efficiency realizations across firms: learning that one merger \(M_k\) can achieve large synergies would imply that other mergers are unlikely to achieve significant synergies.

\(^7\)Similar to Bernheim and Whinston’s (1996) menu auction, firms \(i \neq 0\) make payments even when they are not party to a merger.

\(^8\)To see that firm 0 wants to propose merger \(M^* (\mathcal{F}, \mathcal{A})\), note that using this type of mechanism its optimal merger proposal solves \(\max_{M_k \in (\mathcal{F} \cap \mathcal{A})} \sum_{i \in \mathcal{N} \setminus \{0\}} \pi_i(M_k) - \sum_{i \neq 0} \pi_i(M)\) [which is equivalent to \(\max_{M_k \in (\mathcal{F} \cap \mathcal{A})} \sum_{i \in \mathcal{N} \setminus \{0\}} \pi_i(M_k)\)], provided this exceeds \(\sum_{i \in \mathcal{N}} \pi^0_i - \sum_{i \neq 0} \pi_i(M)\), and is \(M^*\) otherwise.
firms bid $\min_k[\pi_k(M_k) - \pi_k(M_{-k})]$ and the firm whose merger maximizes aggregate profit wins. Moreover, this same firm wins in all Nash equilibria.

With efficient bargaining, Monotonicity holds whenever pre-merger cost differences are small enough that the sum of the pre-merger market share of firms 0 and $k$ weakly exceeds the pre-merger share of any other firm, i.e., $s^0_0 + s^0_k \geq \max_{j \neq 0, k} s^0_j$. To see why, note that multiplying the post-merger first-order condition for each firm $j$ under merger $M_k$ [condition (2) or (3)] by $q_j(M_k)$ and summing over $j$ yields

$$\sum_{i \in \mathcal{N}\backslash \{0\}} \pi_i(M_k) = |Q(M_k)^2 P'(Q(M_k))| H(M_k),$$

where $H(M_k) \equiv \sum_{i \in \mathcal{N}\backslash \{0\}} (s_i(M_k))^2$ is the post-merger industry Herfindahl index. Assumption 1 ensures that the first term, $|Q^2 P'(Q)|$, is increasing in $Q$. By Lemma 2, a reduction in post-merger marginal cost $\tau_k$ leads to a larger $Q(M_k)$, so $\Delta \Pi I(M_k)$ is decreasing in $\tau_k$ if reducing the merged firm’s marginal cost $\tau_k$ induces an increase in $H(M_k)$. Under Assumption 1, a decrease in $\tau_k$ increases the share of the merged firm and decreases the share of every other firm. Since $s^0_0 + s^0_k \geq \max_{j \neq 0, k} s^0_j$ implies $s_k(M_k) \geq \max_{j \neq 0, k} s_j(M_k)$ for any CS-nondecreasing merger $M_k$, this induced change in market shares increases the post-merger Herfindahl index $H(M_k)$ (see Lemma 5 in the Online Appendix). Thus, $\Delta \Pi I(M_k)$ is decreasing in $\tau_k$ if pre-merger cost differences are small enough, while $\Delta CS(M_k)$ is always decreasing in $\tau_k$ by Lemma 2.

Willingness to Propose holds because, by Lemma 1, a CS-neutral merger $M_k$ raises not only the bilateral profit of the merger partners but also aggregate profit.

Finally, Lemma 4 in the Appendix shows that the Ordered Bias condition is also satisfied in this case: if two mergers, $M_j$ and $M_k$ with $k > j$, induce the same non-negative change in consumer surplus $|\Delta CS(M_j) = \Delta CS(M_k) \geq 0|$, then the larger merger $M_k$ induces a greater increase in aggregate profit: $\Delta \Pi I(M_k) > \Delta \Pi I(M_j)$.\footnote{In the Online Appendix we also show that when the bargaining outcome maximizes the joint profit of a subset of firms containing at least all the firms in $\mathcal{K}$, then these conditions hold as well. For example, this would be the case if $N > K = 2$ and the two firms in $\mathcal{K}$ engage in a first-price sealed bid auction for firm 0.}

**Aggregate Surplus Standard.** Suppose that bargaining is efficient and the antitrust authority’s welfare criterion is a weighted average of consumer surplus and aggregate surplus, so that the welfare change from merger $M_k$ is

$$\Delta W(M_k) \equiv (1 - \lambda) \Delta CS(M_k) + \lambda \Delta AS(M_k) = \Delta CS(M_k) + \lambda \Delta \Pi I(M_k),$$

where $\lambda \in [0, 1]$ and $\Delta AS(M_k) \equiv \Delta CS(M_k) + \Delta \Pi I(M_k)$ is the change in aggregate surplus.

Consider, first, Monotonicity. In general, neither $\Delta W$ nor $\Delta \Pi I$ need increase when a firm’s cost decreases. However, both must be decreasing in $\tau_k$ for mergers that are W-nondecreasing if pre-merger marginal cost differences are sufficiently small. To see this, note first that if $\Delta W(M_k) \geq 0$ then $\Delta AS(M_k) > 0$: if $\Delta CS(M_k) < 0$ this follows immediately from (11), while if $\Delta CS(M_k) \geq 0$ then we know from the discussion of efficient bargaining that $\Delta \Pi I(M_k) > 0$, implying again that $\Delta AS(M_k) > 0$.\footnote{In the Online Appendix we also show that when the bargaining outcome maximizes the joint profit of a subset of firms containing at least all the firms in $\mathcal{K}$, then these conditions hold as well. For example, this would be the case if $N > K = 2$ and the two firms in $\mathcal{K}$ engage in a first-price sealed bid auction for firm 0.}
Now consider the extreme case where all firms have the same pre-merger marginal cost $c$. Then, for merger $M_k$ to be W-nondecreasing (and, hence, AS-increasing), it must involve synergies: i.e., we must have $\tau_k < c$.\(^{10}\) Hence, if $M_k$ is W-nondecreasing, then after merger $M_k$ the merged firm is the firm with the lowest marginal cost. Reducing the merged firm’s marginal cost $\tau_k$ therefore increases both aggregate output $Q$ (thereby raising $|Q^2P'(Q)|$) and the Herfindahl index $H$, which from equation (10) increases $\Delta \Pi_I(M_k)$. Moreover, since $\Delta CS(M_k)$ increases, so does $\Delta W(M_k)$. By continuity of consumer surplus and aggregate industry profit in marginal costs, it also follows that if pre-merger marginal cost differences are sufficiently small, then $\Delta W(M_k) \geq 0$ implies that both $\Delta \Pi_I(M_k)$ and $\Delta W(M_k)$ are decreasing in $\tau_k$.

Willingness to Propose holds for small enough cost differences because, in that case, if $\Delta W(M_k) = 0$ then $\Delta CS(M_k) < 0$ and $\Delta AS(M_k) > 0$, which implies that $\Delta \Pi_I(M_k) > 0$.

Finally, Ordered Bias holds as well: Suppose that two W-nondecreasing mergers, $M_j$ and $M_k$ with $k > j$, induce the same change in $W$ but the smaller merger $j$ has the weakly larger aggregate profit level, $\Delta \Pi_I(M_j) \geq \Delta \Pi_I(M_k)$. Lemma 4 (referred to above, and proven in the Appendix) would then imply that $\Delta CS(M_j) > \Delta CS(M_k)$. But then $\Delta W(M_j) > \Delta W(M_k)$, yielding a contradiction. Thus, we must have $\Delta \Pi_I(M_k) > \Delta \Pi_I(M_j)$.

**Differentiated Price Competition.** In the Online Appendix we show that these conditions hold as well when firms instead compete in prices and each initially produce a single symmetrically differentiated good with consumers having CES or multinomial logit demand, provided that the antitrust authority has a consumer surplus standard and bargaining is efficient. The argument relies on the fact that, like the Cournot model, both of these differentiated price competition models are “aggregative games” [Corchon (1994)]. Moreover, if instead bargaining takes the form of the offer game, then the conditions hold provided that the potential consumer surplus gains are not too large. This follows because for CS-neutral mergers the change in aggregate profit equals the change in the bilateral profit of the merging firms (nonmerging firms are unaffected), so close enough to the horizontal axis in Figure 1 the Willingness to Propose and Ordered Bias conditions must hold (Monotonicity is always satisfied).

### 4 Optimal Merger Policy

We can now turn to the optimal policy of the antitrust authority. Recall that the antitrust authority can without loss restrict itself to approval sets in which the set of acceptable cost levels for a merger between firm 0 and each firm $k$, $A_k \subseteq [l, h_k]$, is a union of closed intervals with nonempty interiors.

\(^{10}\)To see this, suppose otherwise that $\tau_k \geq c$. We can decompose the induced change in market structure into two steps: (i) a move from $N + 1$ to $N$ firms, each with marginal cost $c$, and (ii) an increase in the marginal cost of one firm from $c$ to $\tau_k \geq c$. Step (i) induces a reduction in aggregate output but does not affect average production costs, so it reduces aggregate surplus. Step (ii) weakly reduces aggregate output and weakly increases average costs in the industry, so it weakly reduces aggregate surplus. Since aggregate surplus declines, so must $W(M_k)$ – a contradiction to the assumption that merger $M_k$ is W-nondecreasing.
Throughout we restrict attention to such policies.\textsuperscript{11} Let \( \pi_k \equiv \max\{\tau_k : \tau_k \in A_k\} \) denote the largest allowable post-merger cost level for a merger (i.e., the “marginal merger”) between firms 0 and \( k \). Also let \( \Delta CS_k = \Delta CS(k, \pi_k) \) and \( \Delta \Pi_k = \Delta \Pi(k, \pi_k) \) denote the changes in consumer surplus and bilateral profit, respectively, induced by that marginal merger. These are the lowest levels of consumer surplus and bilateral profit in any allowable merger between firms 0 and \( k \). Note that \( \pi_k \) (and thus \( \Delta CS_k \) and \( \Delta \Pi_k \)) is defined only if merger \( M_k \) is approved with positive probability, i.e., only if \( A_k \neq \emptyset \).

At first glance, one may be tempted to conjecture that the antitrust authority can achieve its goal by simply approving any proposed merger that is CS-nondecreasing, i.e., for every \( k \geq 1 \), setting \( A_k = [l, \pi_k] \), where \( \pi_k \) is such that \( \Delta CS(k, \pi_k) = 0 \). Figure 2(a) illustrates such a policy (with heavy trace) for a case in which \( K = 4 \). In fact, this is not an optimal policy. To see this, suppose the antitrust authority instead adopts an approval policy \( A' \) that imposes a slightly tougher standard on the largest merger: setting \( A'_k = A_k \) for each merger \( k < 4 \), and setting \( A'_4 = \{M_4 : \Delta CS(M_4) \geq \varepsilon\} \) for \( \varepsilon > 0 \) sufficiently small. This acceptance set is shown in Figure 2(b). The two policies differ only in the event that the most profitable feasible and acceptable merger under approval policy \( A \), \( M^* (\mathcal{F}, A) \), lies in set \( A \setminus A' \), i.e., only when \( M^* (\mathcal{F}, A) = M_4 \) and \( \Delta CS(M_4) \in [0, \varepsilon] \). Conditional on this event, the expected change in consumer surplus under approval policy \( A \) is bounded from above by \( \varepsilon \), which approaches zero as \( \varepsilon \) becomes small. Under the alternative approval policy \( A' \), and conditioning on the same event, the firms will propose the next-most profitable acceptable merger (which must involve a target \( k < 4 \) or no merger). Since the two policies do not differ in their acceptance sets for such smaller mergers, the expected change in consumer surplus under \( A' \) thus converges to \( E_{\mathcal{F}} [\Delta CS (M^* (\mathcal{F} \setminus M_4, A)) | \Delta \Pi (M^* (\mathcal{F} \setminus M_4, A)) \leq \Delta \Pi_4] > 0 \) as \( \varepsilon \) becomes small.\textsuperscript{12} Hence, the expected change in consumer surplus is larger under \( A' \) than under the naive approval policy \( A \).

Since the naive policy of approving any CS-nondecreasing merger is not optimal, how should the antitrust authority construct its approval policy to maximize expected consumer surplus? Our main result is the following:

**Proposition 1.** Any optimal approval policy \( A \) approves the smallest merger if and only if it is CS-nondecreasing, approves only mergers \( k \in K^+ \equiv \{1, ..., \hat{K}\} \) with positive probability (\( \hat{K} \) may equal \( K \)), and satisfies \( 0 = \Delta CS_1 < \Delta CS_2 < \cdots < \Delta CS_{\hat{K}} \) for all \( k \leq \hat{K} \). That is, the lowest level of consumer surplus change that is acceptable to the antitrust authority equals zero for the smallest merger \( M_1 \), is strictly positive for every other merger \( M_k \), and is monotonically increasing in the size of the merger, while the largest merger(s) may never be approved.

According to Lemma 3, there is a systematic misalignment between firms’ proposal incentives and the interests of the antitrust authority: firms have an incentive to propose a merger that is larger (in

\textsuperscript{11}Thus, when we state that any optimal policy must have a particular form, we mean any optimal interval policy of this sort. There are other optimal policies that add or subtract in addition some measure zero sets of mergers, since these have no effect on expected consumer surplus.

\textsuperscript{12}Recall that \( \Delta \Pi (M^\circ) \equiv 0 \). Hence, \( \Delta \Pi (M^\ast (\mathcal{F} \setminus M_4, A)) = 0 \) if \( M^\ast (\mathcal{F} \setminus M_4, A) = M^\circ \).
Step 1. Observe, first, that the optimal policy $A$ does not approve CS-decreasing mergers: removing all CS-decreasing mergers from the approval set assures that consumer surplus does not decline, without changing the outcome when none of those CS-decreasing mergers would have been proposed.

Step 2. Next, note that every CS-nondecreasing smallest merger ($M_1$) must be included in the optimal approval set. If not, as in the set $A$ depicted in heavy trace in Figure 3(a), we could change the approval set $A$ by adding all CS-nondecreasing mergers $M_1$, resulting in the alternative approval set $A'$ depicted in heavy trace in Figure 3(b). This change of approval sets matters only in the event in which, under $A'$, a CS-nondecreasing merger $M_1$ would be proposed and approved while, under $A$, this merger would not be approved resulting instead in the next-most profitable (in terms of bilateral profit) allowable merger, which may be the null merger $M^\epsilon$.\footnote{Henceforth, whenever we refer to the “next-most profitable allowable (or ‘acceptable’) merger” we mean to include the possibility that this would be the null merger $M^\epsilon$, which has $\Delta \Pi(M^\epsilon) \equiv 0$.} As we have already noted, this next-most profitable allowable merger must increase consumer surplus by less than merger $M_1$ (see Corollary 1 in...
Figure 3: Changing the approval set $\mathcal{A}$ by approving the smallest merger $M_1$ whenever it does not reduce consumer surplus, resulting in approval set $\mathcal{A}'$, raises expected consumer surplus.

the Appendix). Hence, expected consumer surplus is higher under the alternative approval set $\mathcal{A}'$ than under $\mathcal{A}$.

**Step 3.** In any optimal approval set $\mathcal{A}$, the consumer surplus level of the marginal merger $M_k = (k, \bar{c}_k)$, $k \in \mathcal{K}^+$, equals the expected CS-level of the next-most profitable acceptable merger, which we write as

$$E_{\mathcal{A}}^k(\bar{c}_k) \equiv E_{\mathcal{A}}[\Delta CS(M^*(\mathcal{F}\backslash M_k, \mathcal{A})) | M_k = (k, \bar{c}_k) \text{ and } M_k = M^*(\mathcal{F}, \mathcal{A})]$$

$$= E_{\mathcal{A}}[\Delta CS(M^*(\mathcal{F}\backslash M_k, \mathcal{A})) | M_k = (k, \bar{c}_k) \text{ and } \Delta \Pi(M^*(\mathcal{F}\backslash M_k, \mathcal{A})) \leq \Delta \Pi(M_k)].$$

That is, $\Delta CS_k = E_{\mathcal{A}}^k(\bar{c}_k)$ for all $k \in \mathcal{K}^+$. For example, in Figure 4, $\Delta CS_k$ must equal $E_{\mathcal{A}}^k(\bar{c}_2)$, which is the expected level of $\Delta CS$ conditional on the next-most-profitable merger being in the shaded region. To see why this indifference condition must hold, suppose first that the consumer surplus level of the marginal merger $M_k$ is less than the expected CS-level of the next-most profitable acceptable merger, i.e., $\Delta CS_k < E_{\mathcal{A}}^k(\bar{c}_k)$. Consider changing the approval set $\mathcal{A}$ by removing all mergers $M_k$ with $\bar{c}_k \in (\bar{c}_k - \varepsilon, \bar{c}_k]$, thereby increasing $\Delta CS_k$. For $\varepsilon > 0$ sufficiently small, this change in the approval set increases expected consumer surplus.\(^{14}\) Similarly, if $\Delta CS_k > E_{\mathcal{A}}^k(\bar{c}_k)$, the antitrust authority can increase expected consumer surplus by adding to the approval set all mergers $M_k \in (\bar{a}_k, \bar{a}_k + \varepsilon)$ for $\varepsilon > 0$ sufficiently small.\(^{15}\)

\(^{14}\)Note that $k \in \mathcal{K}^+$ implies that $\bar{a}_k > l$, so that $\bar{a}_k - \varepsilon > l$ for $\varepsilon > 0$ sufficiently small.

\(^{15}\)By Step 1 and Assumption 2, we have $\bar{a}_k < h_k$, implying that $\bar{a}_k + \varepsilon < h_k$ for $\varepsilon > 0$ sufficiently small.
Figure 4: The optimal approval policy is such that the increase in consumer surplus induced by the marginal merger \(M_k\) (shown here as \(\Delta CS_2\) for \(k = 2\)) equals the expected consumer surplus change from the next-most-profitable acceptable merger, conditional on the marginal merger being the most profitable merger in the set of feasible and acceptable mergers. The next-most-profitable acceptable merger must therefore lie in the shaded region.

**Step 4.** Next, we can see that any optimal approval policy \(A\) has the property that the increase in bilateral profit induced by a marginal merger is at least as large for larger mergers: that is, \(\Delta \Pi_j \leq \Delta \Pi_k\) for \(j < k\), \(j, k \in K^+\). Panel (a) of Figure 5, where \(\Delta \Pi_2 > \Delta \Pi_3\), depicts a situation where this property is not satisfied. Intuitively, the merger \(\widehat{M}_2\) directly above the marginal merger \((3, \overline{a}_3)\), has a higher level of \(\Delta CS\) than does \((3, \overline{a}_3)\), while resulting in the same expected \(\Delta CS\) if it is rejected. Hence, if \((3, \overline{a}_3)\) is approved, so should be \(\widehat{M}_2\), or more precisely, so should those in the set \(\overline{A}_2^\varepsilon\) (for small \(\varepsilon\)) shown in Figure 5(b).

**Step 5.** Next, we can show that in any optimal approval policy \(A\), the consumer surplus increase induced by the marginal merger is strictly greater for larger mergers, i.e., \(\Delta CS_j < \Delta CS_k\) for \(j < k\), \(j, k \in K^+\). A situation in which this is not true is illustrated in Figure 6, where \(\Delta CS_2 \geq \Delta CS_3\). By the indifference condition of Step 3, \(\Delta CS_3\) must equal the expected \(\Delta CS\) of the next-most profitable allowable merger, i.e., \(\Delta CS_3 = \bar{E}_3^A(\overline{a}_3)\). Now, this expectation is the weighted average of the expected \(\Delta CS\) in two events. First, the next-most profitable allowable merger, say \(M'\), may be more profitable than the marginal merger \((2, \overline{a}_2)\), i.e., \(\Delta \Pi(M') \in [\Delta \Pi_2, \Delta \Pi_3]\). In this event, \(M'\) must (by Step 4) involve a smaller target (either firm 1 or 2). Hence, the expected \(\Delta CS\) in this event strictly exceeds \(\Delta CS_2\). Second, the next-most profitable acceptable merger \(M'\) may be less profitable than the marginal merger \((2, \overline{a}_2)\), i.e., \(\Delta \Pi(M') < \Delta \Pi_2\). By the indifference condition of Step 4, the expected \(\Delta CS\) in this event equals \(\Delta CS_2\). Taking the weighted average of these two events, we conclude that
Figure 5: Panel (a) shows a situation where $\Delta \Pi_k$ is not increasing in $k$; panel (b) shows an improvement in the approval set.

$$\Delta CS_3 = E_3^A(\pi_3) > \Delta CS_2,$$ a contradiction.

Step 6. Finally, we argue that if there is a merger $M_j$ that will never be approved under the optimal policy $\mathcal{A}$, then no larger merger $M_k$, $k > j$, will ever be approved either: that is, $k \notin K^+$ implies $k + 1 \notin K^+$. The result follows by observing that the sum of marginal costs after merger $(3, l)$ is lower than that after merger $(4, l)$, which implies [by (7)] that the largest possible improvement in consumer surplus under merger $M_k$, $\Delta CS(k, l)$, is decreasing in $k$ (as shown in all of the figures). By arguments similar to those showing the monotonicity of $\Delta CS_k$ in $k$ for $k \in K^+$, this implies that if merger $M_k$ is never approved, then neither is any merger that is larger than $M_k$.

**Remark 1.** In our analysis, we have assumed that the support of the post-merger marginal cost $\tau_k$ is given by $[l, h_k]$. That is, the lower bound on $\tau_k$, denoted $l$, is the same for all mergers $M_k$. In the proof of Proposition 1, this assumption was used only in the last step to prove that if merger $M_j$ is never approved, then a larger merger $M_k$, $k > j$, will never be approved either. If we allow for a merger-specific lower bound $l_k$, Proposition 1 continues to hold as long as $\Delta CS(k, l_k) > \Delta CS(k+1, l_{k+1})$ for all $1 \leq k < K$ (which is implied by, but does not imply, $l_k \leq l_{k+1}$). In the general case where no restrictions on the merger-specific lower bounds are imposed, the main conclusion of Proposition 1 carries over: In the optimal approval policy, the smallest merger $M_1$ is approved if and only if it is CS-nondecreasing, while the minimum $\Delta CS$ necessary for any larger merger to be accepted is strictly positive and greater for larger mergers, i.e., $0 = \Delta CS_1 < \Delta CS_j < \Delta CS_k$ for any mergers $j, k \in K^+ \setminus \{1\}$ for which $j < k$. 

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Moreover, if merger $M_j$ is never approved, while the larger merger $M_k$ having $k > j$ is approved with positive probability, then the maximum possible CS-increase induced by merger $M_j$ is less than the minimum CS-increase necessary for approval of the larger merger $M_k$. That is, if $j \notin K^+$ and $k \in K^+$ for $j < k$, then $\Delta CS(j, l_j) < \Delta CS_k$.

5 Cutoff Policies

Proposition 1 shows that in any optimal policy the least efficient acceptable merger involving a target $k$ [the marginal merger $M_k = (k, \pi_k)$] involves a larger increase in consumer surplus (and larger increase in bilateral profit) the larger is the target. Moreover, the result holds for any distributions of post-merger marginal costs. However, it does not fully characterize those marginal mergers. Indeed, while we know that the marginal merger $M_k = (k, \pi_k)$ satisfies the indifference condition $\Delta CS(M_k) = E_k^A(\pi_k)$, the expectation $E_k^A(\pi_k)$ depends on the acceptance sets for mergers other than $k$ (i.e., on $A_j$, $j \neq k$), whose optimal forms depend in turn on merger $k$’s acceptance set $A_k$.

Identifying the marginal merger for each target would be much simpler if we knew that the optimal policy had a “cutoff” structure, in which, for each target $k$, any mergers with greater efficiencies than the marginal merger are accepted. Specifically, a cutoff policy $A^C$ is defined by a set of marginal cost cutoffs, $(\pi_1^C, ..., \pi_K^C)$, such that $M_k = (k, \pi_k) \in A^C$ if and only if $\pi_k \leq \pi_k^C$. In that case, Proposition 1 would imply that the marginal mergers could be found by a simple recursive procedure: accept all CS-
nondecreasing mergers $M_k$ [i.e., set $\pi^C_k = \tilde{c}_1(Q^P)$], then for $k = 2, \ldots, K$ recursively identify the largest post-merger cost level $\bar{\pi}^C_k$ for which $\Delta CS_k(k, \bar{\pi}^C_k) = E^A_k(\bar{\pi}^C_k)$, where now the expectation $E^A_k(\bar{\pi}^C_k)$ depends only on the already-determined cutoffs for mergers $M_1, \ldots, M_{k-1}$. If $\Delta CS(k, \bar{\tau}_k) < E^A_k(\bar{\tau}_k)$ for all $\bar{\tau}_k \in [l, h_k]$, then no such cutoff exists for merger $M_k$, so that $A^C_k = \emptyset$. Moreover, this will also imply that $A^C_k = \emptyset$ for all $k' > k$.

Unfortunately, however, as the following example illustrates, the optimal policy need not have a cutoff structure.

Example 1. Consider a four-firm industry (so $N = 3$) in which industry inverse demand is $P(Q) = 1 - Q$. Pre-merger marginal costs are $c_0 = c_2 = 0.5$, $c_1 = 0.55$, and $c_3 = 0.45$, so the pre-merger market shares are $s_0 = s_2 = 1/4$, $s_1 = 1/8$, and $s_4 = 3/8$. Pre-merger consumer surplus is 0.8. Firm 0 can merge with either firm 1 or firm 2 (so $K = 2$). Each of these mergers is always feasible; i.e., $\theta_1 = \theta_2 = 1$. Merger $M_2$ results in a post merger marginal cost that has a continuous density on $[0.5, 0.2]$, while merger $M_1$’s post-merger marginal cost is 0.4 with probability 0.1 and 0.3 with probability 0.9.\textsuperscript{16} For these two mergers $M_1$, $(\Delta \Pi(1, 0.4), \Delta CS(1, 0.4)) = (0.0227, 0.0051)$ and $(\Delta \Pi(1, 0.3), \Delta CS(1, 0.3)) = (0.0564, 0.0157)$, and their unconditional expected $\Delta CS$ is 0.0146. It is straightforward to verify that, in this case, the optimal approval policy $A^*$ is such that $A^*_1 = \{0.3, 0.40\}$ and $A^*_2 = \{0.2, 0.260\} \cup [0.298, 0.391]$ where $\Delta CS(2, 0.391) = 0.0051$, $\Delta CS(2, 0.260) = 0.0146$, and $\Delta \Pi(2, 0.298) = 0.0564$. This situation is illustrated in Figure 7. To see why the optimal approval policy for $M_2$ does not have a cut-off structure, note that for any post-merger marginal cost $\bar{\tau}_2 \in (0.260, 0.298)$, $M_2$ would always be the proposed merger if it were approved when proposed. But the induced change in consumer surplus from $M_2$ would be less than 0.0146, the expected $\Delta CS$ from $M_1$. However, once $\bar{\tau}_2$ rises just above 0.298, merger $M_2$ would only be proposed if $M_1 = (1, 0.4)$, so the expected $\Delta CS$ from $M_1$ conditional on the fact that merger $(2, \bar{\tau}_2)$ was proposed falls to 0.0051 and it is optimal to accept $M_2$. This remains true until $\bar{\tau}_2$ falls to 0.391, where $\Delta CS(2, 0.391) = 0.0051$.

Nonetheless, our next result provides a sufficient condition that ensures that the recursively-defined cutoff policy is in fact optimal. To proceed, let $A^C(J) \subseteq \Pi_{k \in J}[l, h_k]$ denote the recursively-defined cutoff policy when only mergers with targets in set $J \subseteq K$ are possible; that is, when we suppose that there is no possibility for a merger with any target $k \notin J$. [The policy $A^C(J)$ specifies $\#J$ cutoffs.] For convenience, when $J = K$ we write $A^C \equiv A^C(K)$. We also let $\bar{\pi}^C_k(J)$ denote the cutoff level of marginal cost for a merger with target $k$ in cutoff policy $A^C(J)$ (defined for mergers accepted with positive probability).

In addition, for a set of targets $J \subseteq K$, define the realized set of feasible mergers to be $\mathcal{F}_J$, and

\textsuperscript{16}For simplicity, the example considers a case where, contrary to the assumptions of our model, one of the mergers has a finite support of post-merger marginal costs, and our assumptions about the upper and lower bounds of the firms’ post-merger cost distributions do not hold. But the same insight would obtain if we perturbed the example so that these assumptions were satisfied. The lower bound for $\tau_2$ insures that all firms remain active after merger $M_2$.  

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Figure 7: The optimal merger approval policy in Example 1, which is not a cutoff policy.

[recalling that \( \Delta \Pi(M^\circ) \equiv 0 \)] define the function

\[
ECS(\Delta \Pi; A, J) \equiv E_{\tilde{\Pi}}[\Delta CS(M^\circ(\tilde{\Pi}, A))] | \Delta \Pi(M^\circ(\tilde{\Pi}, A)) \leq \Delta \Pi
\]

as the expected value of \( \Delta CS \) under policy \( A \subseteq \Pi_{k \in J} [l, h_k] \) from the most profitable acceptable merger involving targets in set \( J \), conditional on that merger’s increase in bilateral profit being no greater than \( \Delta \Pi \).\(^{17}\) Note that the structure of \( A \) at profit levels above \( \Delta \Pi \) affects the value of this conditional expectation by changing the conditional distributions of post-merger marginal costs. Specifically, the probability of some merger \( M_j \) in set \( \mathcal{M}_J \) \( \subseteq \{M_j : \Delta \Pi(M_j) \leq \Delta \Pi \} \) being feasible conditional on the most profitable acceptable merger having a profit level below \( \Delta \Pi \) is \( \Pr(M_j \in \mathcal{M}_J)/(1 - \Pr(\Delta \Pi(M_j) > \Delta \Pi) \text{ and } M_j \in A_j) \).

We then have the following result (whose proof is in the Appendix):

**Proposition 2.** Suppose that for every \( J \subseteq \mathcal{K} \) with \( 1 \in J \) the following property holds:\(^{18}\)

\[
\text{Every merger } M_k = (k, \tau_k) \in \mathcal{A}^C(J) \text{ has } \Delta CS(M_k) > ECS(\Delta \Pi(M_k); \mathcal{A}^C(J \setminus k), J \setminus k). \tag{12}
\]

Then, the cutoff policy \( \mathcal{A}^C \) is an optimal policy.

\(^{17}\) Thus, \( E_{\Pi}^k(\tau_k) = ECS(\Delta \Pi(k, \tau_k); \mathcal{A}_{\mathcal{K}\setminus k}, \mathcal{K}\setminus k) \) where \( \mathcal{A}_{\mathcal{K}\setminus k} \equiv \Pi_{j \in \mathcal{K}\setminus k} A_j \).

\(^{18}\) Property (12) necessarily holds for \( j = 1 \); the assumption made here is that it holds for all \( j > 1 \).
While Proposition 2 does not offer a condition on primitives, it allows us to verify that the recursively-derived cutoff policy is optimal. The result says that this cutoff policy is an optimal policy provided that the consumer surplus change $\Delta CS(M_k)$ of each merger $M_k$ it approves exceeds the expected $\Delta CS$ of the next most profitable acceptable merger in policy $\mathcal{A}^C(J\setminus k)$, the recursively-defined cutoff policy for each possible set of alternative mergers $J\setminus k$ that includes merger $M_1$. The following example illustrates its use.

**Example 2.** Consider the same four-firm industry as in Example 1, but now firm 0 can merge with each of the other firms (so $K = N = 3$). For this industry, the naive policy marginal cost cutoffs (where any CS-nondecreasing merger is accepted) are $\pi_1^N = 0.45$, $\pi_2^N = 0.40$, $\pi_3^N = 0.35$. Suppose that each merger has a $3/4$ probability of being feasible (so $\theta_k = 0.75$ for $k = 1, 2, 3$) and that, conditional on being feasible, the post-merger marginal cost is distributed with a beta distribution with parameters $\beta = 1$ and $\alpha = 5$ and support between the merger’s naive cutoff and 0.2.\(^{19}\) With these distributions, the merger process would increase expected consumer surplus by 6.44% if there were there no informational asymmetry between the firms and the antitrust authority (so that whichever CS-nondecreasing merger most increased consumer surplus would always be implemented). The cutoffs in the recursively-defined cutoff policy are $\pi_1 = 0.45$, $\pi_2 = 0.383$, and $\pi_3 = 0.316$, with associated changes in consumer surplus of $\Delta CS_1 = 0$, $\Delta CS_2 = 0.00170$, and $\Delta CS_3 = 0.00346$. This policy achieves 90.30% of the first-best increase in expected consumer surplus, while the naive policy which accepts all CS-nondecreasing mergers achieves 79.83% of this amount. One can verify (through computation) that the sufficient condition of Proposition 2 is satisfied in this case. For example, to check condition (12) for merger $M_2$, first for $J/k = \{1\}$ and then for $J/k = \{1, 3\}$, we need to compare the consumer surplus level $\Delta CS(M_2)$ for each merger $M_2$ with $\tau_2 < 0.383$ to the conditional expectation $ECS(\Delta \Pi(M_2); \mathcal{A}^C(J/k))$, $\{J/k\}$ of the consumer surplus in the next-most-profitable merger that is acceptable in the recursively-defined cutoff policy $\mathcal{A}^C(J/k)$. For $J/k = \{1, 3\}$, the policy $\mathcal{A}^C(\{1, 3\})$ has cut-off levels $\pi_1^C(\{1, 3\}) = 0.45$ and $\pi_3^C(\{1, 3\}) = 0.323$. Looking at two specific mergers $M_2$ in this case, for $M_2 = (2, 0.35)$ we have $\Delta CS(M_2) = 0.0051$ and $ECS(\Delta \Pi(M_2); \mathcal{A}^C(\{1, 3\}), \{1, 3\}) = 0.0025$, while for $M_2 = (2, 0.3)$ we have $\Delta CS(M_2) = 0.0103$ and $ECS(\Delta \Pi(M_2); \mathcal{A}^C(\{1, 3\}), \{1, 3\}) = 0.0034$\(^{20}\) In both cases, condition (12) is satisfied.

When cutoff rules are optimal we can also explore how changes in underlying parameters alter the nature of the optimal policy. Here we provide a result (proof in the Appendix) on the effect of changes in the merger feasibility probabilities $\theta_k$ on the optimal policy, assuming that the optimal policy has

\(^{19}\)One can think of this situation as having a $1/4$ probability of there being no CS-increasing merger, and a $3/4$ probability of a CS-increasing merger. The beta distribution has a pdf $f(x|\alpha, \beta)$ that is proportional to $x^{\alpha-1}(1 - x)^{\beta-1}$. Its mean is the lower bound of its support plus a fraction $\alpha/\alpha + \beta$ of the difference between its support’s upper and lower bounds. When $\beta = 1$ and $\alpha = 5$, the pdf is an increasing function, so that small efficiency gains are more likely than large ones. The lower bound of $l = 0.2$ ensures that all firms remain active after any merger.

\(^{20}\)For post-merger costs $\tau_2 > 0.339$, the next-most-profitable acceptable merger in $\mathcal{A}^C(\{1, 3\})$ must be $M_1$ or no merger, while for $\tau_2 < 0.339$ it could also be $M_3$. 

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a cutoff structure. Intuitively, lower feasibility probabilities should move the optimal policy toward the naive one. For example, as all \( \theta_i \)'s approach zero, the optimal policy approaches the naive policy: since there is almost no chance that two mergers are feasible, firms almost never have a choice of which merger to propose. Our result builds on this intuition:

**Proposition 3.** Consider a decrease in the probability of merger \( M_k \)'s feasibility from \( \theta_k \) to \( \theta_k' < \theta_k \), assuming that \( M_k \) is initially approved with positive probability (i.e., \( k \leq \hat{K} \)). Then, under the optimal merger approval policy, \( \Delta CS_j' = \Delta CS_j \) for any weakly smaller merger \( M_j, j \leq k \), and \( \Delta CS_j' < \Delta CS_j \) for any larger merger \( M_j, j > k \), that is approved with positive probability.

### 6 Conclusion

In this paper, we have analyzed the optimal merger approval policy of an antitrust authority which seeks to maximize expected consumer surplus when there are several mutually exclusive merger possibilities and firms can choose which merger to propose. In our model, there is a single acquirer that can make a merger proposal to one of several, ex ante heterogeneous merger partners. While the feasibility and post-merger marginal costs of the various potential mergers is not known to the antitrust authority, the antitrust authority can observe the characteristics of the proposed merger. We have shown that in this environment the antitrust authority optimally commits to a policy that imposes a tougher standard on mergers involving firms with a larger pre-merger market share, or equivalently in our model, inducing a larger increase in the naively-computed Herfindahl index: the required minimum increase in consumer surplus is greater for mergers that are larger in this sense. The form of this optimal policy is a response to a fundamental bias that we have shown exists in firms’ proposal incentives: larger mergers are sometimes proposed when smaller ones that would lead to greater increases in consumer surplus are available, while the reverse never happens. The optimal policy therefore rejects some consumer surplus-enhancing larger mergers to induce firms to propose better smaller ones.

Our model and result also suggest some interesting questions for future research. For example, while in our model pre-merger marginal costs are taken to be exogenous, in practice they are likely to be the result of investments by the firms. To the extent that merger policy depends on pre-merger costs or market shares, an optimal policy needs to take account of any effects on these investments. A second issue concerns remedies. In practice, many mergers are approved subject to some remedy that is designed to improve consumer welfare, and which presumably lowers the profits of the merged firm. While imposing remedies may improve consumer welfare ex post [shifting the merger to the Northwest in \( (\Delta \Pi, \Delta CS) \)-space], our model suggests that insisting on remedies could also lead to changes in the mergers that firms propose. Given this fact, an interesting question is therefore the extent to which it is optimal to impose remedies as a requirement to gain approval. Finally, analyzing optimal merger policy with merger choice in a setting with a richer structure of merger possibilities than in the present model – in which all mergers involve a single pivotal firm – is an important direction for future work.
7 Appendix

We first state a simple but useful corollary of Lemmas 2 and 3:

**Corollary 1.** If two CS-nondecreasing mergers \( M_j \) and \( M_k \) with \( k > j \) have \( \Delta \Pi(M_k) \leq \Delta \Pi(M_j) \), then \( \Delta CS(M_k) < \Delta CS(M_j) \).

**Proof.** Suppose instead that \( \Delta CS(M_k) \geq \Delta CS(M_j) \). Then there exists a \( \tau'_k > \tau_k \) such that \( \Delta CS(k, \tau'_k) = \Delta CS(M_j) \). But this implies (using Lemma 2 for the first inequality and Lemma 3 for the second) that \( \Delta \Pi(M_k) > \Delta \Pi(k, \tau'_k) > \Delta \Pi(M_j) \), a contradiction. \( \square \)

**Lemma 4.** Suppose two mergers, \( M_j \) and \( M_k \), with \( k > j \), induce the same non-negative change in consumer surplus, \( \Delta CS(M_j) = \Delta CS(M_k) \geq 0 \). Then, the larger merger \( M_k \) induces a greater increase in aggregate profit.

**Proof.** From the discussion in the main text, the post-merger aggregate profit is given by (10). As both mergers induce the same level of consumer surplus (and thus the same \( Q \)), the first term on the right-hand side of (10) is the same for both mergers. It thus suffices to show that the larger merger \( M_k \) induces a larger value of \( H \) than the smaller merger \( M_j \).

Now, as both mergers induce the same \( Q \), Assumption 1 implies that the output of any firm not involved in \( M_j \) or \( M_k \) is the same under both mergers. Hence,

\[
s_k(M_k) + s_j(M_k) = s_k(M_j) + s_j(M_j). \tag{13}
\]

Next, recall that a CS-nondecreasing merger increases the share of the merging firms and reduces the share of all nonmerging firms. Thus, we have \( s_k(M_k) \geq s_k + s_0 > s_k(M_j) \) and \( s_j(M_j) \geq s_j + s_0 > s_j(M_k) \). In addition, since total output is the same after both mergers and \( c_k < c_j \), we also have \( s_j(M_k) < s_k(M_k) \). By (13), this in turn implies that \( s_k(M_k) > s_j(M_j) \). Hence, the distribution of market shares after the larger merger \( M_k \) is a sum-preserving spread of those after the smaller merger \( M_j \):

\[
s_k(M_k) > \max\{s_j(M_j), s_k(M_j)\} \geq \min\{s_j(M_j), s_k(M_j)\} > s_j(M_k). \tag{14}
\]

Given (13), it follows that \( s_k(M_k)^2 + s_j(M_k)^2 > s_k(M_j)^2 + s_j(M_j)^2 \). Since \( s_i(M_k) = s_i(M_j) \) for all \( i \neq 0, j, k \), this implies that \( H \) is larger after \( M_k \) than after \( M_j \). \( \square \)

**Proof of Proposition 1.** The proof proceeds in a number of steps.

**Step 1.** We observe first that an optimal policy does not approve CS-decreasing mergers. That is, \( \Delta CS_k > 0 \) for all \( k \in K^+ \), where \( K^+ \) denotes those targets for whom the probability of having a merger \( M_k \in A \) is strictly positive. To see this, suppose the approval set \( A \) includes CS-decreasing mergers, and consider the set \( A^+ \subseteq A \) that removes any mergers in \( A \) that reduce consumer surplus. Since this change only matters when the bilateral profit-maximizing merger \( M^*(S_A) \) under set \( A \) is no
longer approved under $\mathcal{A}^\dagger$, the change in expected consumer surplus from this change in the approval policy equals $\Pr(M^*(\tilde{\mathcal{A}}^\dagger) \in \mathcal{A}^\dagger \setminus \mathcal{A}^+)$, the probability of this event happening, times the conditional expectation

$$E_{\bar{g}}[\Delta CS(M^*(\tilde{\mathcal{A}}^\dagger)) - \Delta CS(M^*(\tilde{\mathcal{A}})) | M^*(\tilde{\mathcal{A}}) \in \mathcal{A}^\dagger \setminus \mathcal{A}^+] .$$

Since $\Delta CS(M^*(\tilde{\mathcal{A}}^\dagger))$ is necessarily nonnegative by construction of $\mathcal{A}^\dagger$, and $\Delta CS(M^*(\tilde{\mathcal{A}}))$ is strictly negative whenever $M^*(\tilde{\mathcal{A}}) \in \mathcal{A}^\dagger \setminus \mathcal{A}^+$, this change is strictly positive.

**Step 2.** Next, any smallest merger $M_1$ that is CS-nondecreasing must be approved. To see this, suppose that the approval set is $\mathcal{A}$ but that $\mathcal{A} \subset \mathcal{A}' \equiv (\mathcal{A} \cup \{(1, \tau_1) : \Delta CS(1, \tau_1) \geq 0\})$. Figure 3 depicts two such sets, $\mathcal{A}$ and $\mathcal{A}'$. Because a change from $\mathcal{A}'$ to $\mathcal{A}$ matters only when the bilateral profit-maximizing merger $M^*(\tilde{\mathcal{A}}')$ under $\mathcal{A}'$ is no longer approved under $\mathcal{A}$, the change in expected consumer surplus by using $\mathcal{A}'$ rather than $\mathcal{A}$ equals $\Pr(M^*(\tilde{\mathcal{A}}') \in \mathcal{A}' \setminus \mathcal{A})$ times

$$E_{\bar{g}}[\Delta CS(M^*(\tilde{\mathcal{A}}')) - \Delta CS(M^*(\tilde{\mathcal{A}})) | M^*(\tilde{\mathcal{A}}) \in \mathcal{A}' \setminus \mathcal{A}] .$$

(15)

By Corollary 1 and the fact that $\mathcal{A}' \setminus \mathcal{A}$ contains only smallest mergers (between firms 0 and 1), whenever $M^*(\tilde{\mathcal{A}}') \in \mathcal{A}' \setminus \mathcal{A}$ (which implies $\Delta \Pi(M^*(\tilde{\mathcal{A}}')) > \Delta \Pi(M^*(\tilde{\mathcal{A}}))$) we have $\Delta CS(M^*(\tilde{\mathcal{A}}')) > \Delta CS(M^*(\tilde{\mathcal{A}}))$, so (15) is strictly positive. This can be seen in Figure 3, and implies in particular that $\Delta CS_1 = 0$.

**Step 3.** Next, we claim that in any optimal policy, for all $k \in K^+$, $\Delta CS_k$ must equal the expected change in consumer surplus from the next-most-profitable merger (i.e., from the merger with the second-highest bilateral profit change) $M^*(\tilde{\mathcal{A}} \setminus \{k, \bar{\pi}_k\}, \mathcal{A})$, conditional on merger $M_k = (k, \bar{\pi}_k)$ being the most profitable merger in $\tilde{\mathcal{A}} \setminus \mathcal{A}$. Defining the expected change in consumer surplus from the next-most-profitable merger $M^*(\tilde{\mathcal{A}} \setminus M_k, \mathcal{A})$, conditional on merger $M_k = (k, \bar{\pi}_k)$ being the most profitable merger in $\tilde{\mathcal{A}} \setminus \mathcal{A}$, to be

$$E^A_k(\bar{\pi}_k) \equiv E_{\bar{g}}[\Delta CS(M^*(\tilde{\mathcal{A}} \setminus M_k, \mathcal{A})) | M_k = (k, \bar{\pi}_k) \text{ and } M_k = M^*(\tilde{\mathcal{A}}, \mathcal{A})]$$

(16)

$$= E_{\bar{g}}[\Delta CS(M^*(\tilde{\mathcal{A}} \setminus M_k, \mathcal{A})) | M_k = (k, \bar{\pi}_k) \text{ and } \Delta \Pi(M^*(\tilde{\mathcal{A}} \setminus M_k, \mathcal{A})) \leq \Delta \Pi(M_k)] ,$$

(17)

this means that

$$\Delta CS_k = E^A_k(\bar{\pi}_k) .$$

(18)

In Figure 4 the possible locations of the next-most-profitable merger when the most profitable merger is $M_2 = (2, \bar{\pi}_2)$ are shown as a shaded set. The quantity $E^A_2(\bar{\pi}_2)$ is the expectation of the change in consumer surplus for the merger that has the largest change in bilateral profit among mergers other than $M_2$, conditional on all of these other mergers lying in the shaded region of the figure.

To see that (18) must hold for all $k \in K^+$, suppose first that $\Delta CS_k > E^A_k(\bar{\pi}_{k'})$ for some $k' \in K^+$ and consider the alternative approval set $\mathcal{A} \cup \mathcal{A}_{k'}^c$, where

$$\mathcal{A}_{k'}^c \equiv \{ M_k : M_k = (k', \bar{\pi}_{k'}) \text{ with } \bar{\pi}_{k'} \in (\bar{\pi}_{k'}, \bar{\pi}_{k'} + \epsilon) \} .$$

(By Step 1 and Assumption 2, we have $\bar{\pi}_{k'} < h_{k'}$, implying that $\bar{\pi}_{k'} + \epsilon < h_{k'}$ for $\epsilon > 0$ sufficiently small.) For any $\epsilon > 0$, the change in expected consumer surplus from changing the approval set from
\[ A \to A \cup A_{k'} \text{ equals } \Pr(M^*(\tilde{g}, A \cup A_{k'}) \in A_{k'}) \text{ times} \]

\[ E_{\tilde{g}}[\Delta CS(M^*(\tilde{g}, A \cup A_{k'})) - \Delta CS(M^*(\tilde{g}, A))|M^*(\tilde{g}, A \cup A_{k'}) \in A_{k'}]. \tag{19} \]

This conditional expectation can be rewritten as

\[ E_{\tilde{g}}[\Delta CS(M^*(\tilde{g}, A \cup A_{k'})) - E_k^A(\bar{c}_{k'})|M^*(\tilde{g}, A \cup A_{k'}) \in A_{k'}], \tag{20} \]

where \( \bar{c}_{k'} \) is the realized cost level in the bilateral profit-maximizing merger \( M^*(\tilde{g}, A \cup A_{k'}) \), which is a merger of firms 0 and \( k' \) when the conditioning statement is satisfied. By continuity of \( \Delta CS(k', \bar{c}_{k'}) \) and \( E_k^A(\bar{c}_{k'}) \) in \( \bar{c}_{k'} \), there exists an \( \varepsilon > 0 \) such that \( \Delta CS(M_{k'}) > E_k^A(\bar{c}_{k'}) \) for all \( M_{k'} \in A_{k'} \), provided \( \varepsilon \in (0, \bar{\varepsilon}) \). For all such \( \varepsilon \), the conditional expectation (20) is strictly positive so this change in the approval set would strictly increase expected consumer surplus. A similar argument applies if \( \Delta CS_{k'} < E_k^A(\bar{c}_{k'}) \).

**Step 4.** Next, we argue that for all \( j < k \) such that \( j, k \in \mathcal{K}^+ \) it must be that \( \Delta \Pi_{j} \leq \Delta \Pi_{k} \); that is, the bilateral profit change in the marginal merger by target \( j \) must be no greater than the bilateral profit change in the marginal merger by any larger target \( k \). Figure 5(a) shows a situation that violates this condition, where the marginal merger by target 3 causes a smaller bilateral profit change, \( \Delta \Pi_{j} \), than the marginal merger by the smaller target 2, \( \Delta \Pi_{k} \).

For \( j \in \mathcal{K}^+ \), let \( k' = \arg \min_{k \in \mathcal{K}^+, k > j} \Delta \Pi_{k} \) and suppose that \( \Delta \Pi_{k'} < \Delta \Pi_{j} \). We know from the previous step that \( \Delta CS_{k'} = E_k^A(\bar{c}_{k'}) \). Let \( \bar{c}_{j}' \) be the post-merger cost level satisfying \( \Delta \Pi(j, \bar{c}_{j}') = \Delta \Pi_{k'} \), and consider a change in the approval set from \( \mathcal{A} \to \mathcal{A} \cup \mathcal{A}_j' \) where

\[ \mathcal{A}_j' \equiv \{ M_j : M_j = (j, \bar{c}_{j}) \text{ with } \bar{c}_{j} \in (\bar{c}_{j}', \bar{c}_{j}' + \varepsilon) \}. \]

The set \( \mathcal{A}_j' \) is shown in Figure 5(b). The change in expected consumer surplus from this change in the approval set equals \( \Pr(M^*(\tilde{g}, A \cup A_j') \in \mathcal{A}_j') \) times

\[ E_{\tilde{g}}[\Delta CS(M^*(\tilde{g}, A \cup A_j')) - E_j^A(\bar{c}_{j}')|M^*(\tilde{g}, A \cup A_j') \in \mathcal{A}_j'], \tag{21} \]

where \( \bar{c}_{j} \) is the realized cost level in the aggregate profit-maximizing merger \( M^*(\tilde{g}, A \cup A_j') \), which is a merger of firms 0 and \( j \) when the conditioning statement is satisfied. As \( \varepsilon \to 0 \), the expected change in (21) converges to

\[ \Delta CS(j, \bar{c}_{j}') - E_j^A(\bar{c}_{j}') = \Delta CS(j, \bar{c}_{j}') - E_k^A(\bar{c}_{k'}) > \Delta CS_{k'} - E_k^A(\bar{c}_{k'}) = 0, \]

where the inequality follows from Corollary 1 since \( \Delta \Pi(j, \bar{c}_{j}') = \Delta \Pi_{k'} \).

**Step 5.** We next argue that \( \Delta CS_{j} < \Delta CS_{k} \) for all \( j, k \in \mathcal{K}^+ \) with \( j < k \). Suppose otherwise; i.e., for some \( j, h \in \mathcal{K}^+ \) with \( h > j \) we have \( \Delta CS_{j} \geq \Delta CS_{h} \). Define \( k = \arg \min \{ h \in \mathcal{K}^+ : h > j \text{ and } \Delta CS_{j} \geq \Delta CS_{h} \} \).

Figure 6 depicts such a situation where \( j = 2 \) and \( k = 3 \).
By Step 3, we must have $E_j^A(\pi_j) = \Delta CS_j \geq \Delta CS_k = E_k^A(\pi_k)$. But recalling (17), $E_k^A(\pi_k)$ can be written as a weighted average of two conditional expectations:

$$E_\delta[\Delta CS(M^*(\mathcal{F}\setminus M_k, A)) | M_k = (k, \pi_k), M_k = M^*(\mathcal{F}, A), \text{ and } \Delta \Pi(M^*(\mathcal{F}\setminus M_k, A)) < \Delta \Pi_j]$$

(22)

and

$$E_\delta[\Delta CS(M^*(\mathcal{F}\setminus M_k, A)) | M_k = (k, \pi_k), M_k = M^*(\mathcal{F}, A), \text{ and } \Delta \Pi(M^*(\mathcal{F}\setminus M_k, A)) \in [\Delta \Pi_j, \Delta \Pi_k]].$$

(23)

Expectation (22) conditions on the event that the next-most-profitable merger other than $(k, \pi_k)$ induces a bilateral profit change less than $\Delta \Pi_j$, the bilateral profit change of merger $(j, \pi_j)$. Since no merger in $A$ by either target $k$ or $j$ can have such a profit level (since $\Delta \Pi_k \geq \Delta \Pi_j$ by Step 4), the expectation (22) must exactly equal $E_j^A(\pi_j)$. Now consider the expectation (23). If $\Delta \Pi(M^*(\mathcal{F}\setminus M_k, A)) \in [\Delta \Pi_j, \Delta \Pi_k)$, it could be that (i) $M^*(\mathcal{F}\setminus M_k, A) = (j, \pi_j)$ for some $\pi_j < \pi_j$, or (ii) $M^*(\mathcal{F}\setminus M_k, A) = (r, \pi_r)$ for some $r < j$, or (iii) $M^*(\mathcal{F}\setminus M_k, A) = (r, \pi_r)$ for some $r > j$ and $r < k$. Now, in case (i) it is immediate that $\Delta CS(M^*(\mathcal{F}\setminus M_k, A) \geq CS_j$, with strict inequality whenever $\pi_j = \pi_j$. In case (ii), the fact that $\Delta \Pi(r, \pi_r) \geq \Delta \Pi_j$ implies by Corollary 1 that

$$\Delta CS(M^*(\mathcal{F}\setminus M_k, A) = \Delta CS(r, \pi_r) > CS_j = E_j^A(\pi_j).$$

(24)

In case (iii), (24) follows from the definition of $k$. Thus, expectation (23) must strictly exceed $E_j^A(\pi_j)$, which leads to a contradiction.

Step 6. Finally, we argue that $K^+ = \{1, \ldots, \tilde{K}\}$ for some $\tilde{K} \leq K$. To establish this fact, we show that if $k \not\in K^+$ and $k < \tilde{K}$, then $k + 1 \not\in K^+$. As noted in the text, we first observe that $\Delta CS(k, l) > \Delta CS(k + 1, l)$. Thus, if $k + 1 \in K^+$, then $\Delta CS_{k+1} < \Delta CS(k, l)$. The result then follows by an argument similar to that in Step 5: By Step 3, $\Delta CS_{k+1}$ must equal the expected $\Delta CS$ of the next-most-profitable allowable merger, i.e., $\Delta CS_{k+1} = E_{k+1}^A(\pi_{k+1})$. This expectation is the weighted average of the expected $\Delta CS$ in two events: first that the next-most-profitable allowable merger, say $M'$, lies to the right of the best possible merger $M_k$, i.e., $\Delta \Pi(M') \in [\Delta \Pi(k, l), \Delta \Pi_{k+1}$] and, second, that it lies to the left of the best possible merger $M_k$, i.e., $\Delta \Pi(M') < \Delta \Pi(k, l)$. In the first event (which may be empty), the resulting $\Delta CS$ must exceed that of the marginal merger $M_{k+1}$, i.e., $\Delta CS(M') > \Delta CS_{k+1}$. In the second possibility, the expected $\Delta CS$ in that event must weakly exceed $\Delta CS(k, l)$, as otherwise (by an argument like that in Step 3) the expected $\Delta CS$ could be increased by accepting all mergers $(k, \pi_k)$ with $\pi_k \in [l, l + \varepsilon]$, for $\varepsilon > 0$ sufficiently small, in the approval set. Taking the weighted average of these two events, it follows that $\Delta CS_{k+1} = E_{k+1}^A(\pi_{k+1}) \geq \Delta CS(k, l)$, a contradiction. 

Proof of Proposition 2. Denote by $A^*(\Delta \Pi|J)$ a policy that is an element of $\arg\max_{\Delta \Pi \subseteq J} E\mathbb{C}(\Delta \Pi; A, J)$ for a given $J$ and $\Delta \Pi$. Also, define $P(\Delta \Pi|J, A) \equiv \{k \in J : \Delta \Pi(k, \pi_k) < \Delta \Pi\}$ as the set of targets in $J$ who may have an acceptable merger with profit below $\Delta \Pi$ under policy $A \subseteq J$. Note that changes to $A$ that alter acceptance sets only for $k \not\in P(\Delta \Pi|J, A)$ and leave $P(\Delta \Pi|J, A)$ unchanged have no effect on the value of $E\mathbb{C}(\Delta \Pi; A, J)$. Finally, for any set $A$, let $A_J \equiv \Pi_{j \in J} A_J$. 

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With these preliminaries, we now establish the result. Observe first that, for any $J$, a sufficient condition for $M_k$ with $k \in J$ and $\Delta \Pi(M_k) < \Delta \Pi$ to be approved in any solution to $\max_{A \subseteq \Pi \times \Pi_j} \text{ECS}(\Delta \Pi; A, J)$ is that its CS-level, $\Delta CS(M_k)$, strictly exceeds $\max_{A \subseteq \Pi \times \Pi_j} \text{ECS}(\Delta \Pi(M_k); A, J \setminus k)$. We will establish the result through an induction argument that shows that for all $\Delta \Pi$ and any $J$ such that $1 \in J$, if $A^*(\Delta \Pi|J) \in \arg \max_{A \subseteq \Pi \times \Pi_j} \text{ECS}(\Delta \Pi; A, J)$ then

$$P(\Delta \Pi|J, A^*(\Delta \Pi|J)) = P(\Delta \Pi|J, A^C(J))$$

(25)

and

$$A^*_k(\Delta \Pi|J) = A^C_k(J) \text{ for all } k \in P(\Delta \Pi|J, A^C(J)).$$

(26)

That is, any policy $A$ that maximizes $\text{ECS}(\Delta \Pi, A, J)$ accepts with positive probability [conditional on the most profitable acceptable merger having $\Delta \Pi(M_j) \leq \Delta \Pi$] mergers involving the same set of targets as does the cutoff policy $A^C(J)$, and coincides with the cutoff policy $A^C(J)$ for all such targets. In particular, this implies that the cutoff policy $A^C(J) \in \arg \max_{A \subseteq \Pi \times \Pi_j} \text{ECS}(\Delta \Pi; A, J)$ for all $\Delta \Pi$ and any $J$ such that $1 \in J$. Taking $\Delta \Pi = \infty$ and $J = K$ will then yield the result.

Consider first the set $J = \{1\}$. Then, we have $\pi_1^*(J) = \beta_1(Q^\circ)$. Moreover, it is immediate – given our earlier discussion – that (25) and (26) hold for all $\Delta \Pi$.

Now consider any set $J = J_n$ with $\#J_n = n$ and $1 \in J_n$, and assume:

**Induction Hypothesis 1:** Properties (25) and (26) hold for any set $J = J_{n'}$ with $1 \in J_{n'}$ and $n' < n$.

Number the targets in set $J_n$ in increasing order of their pre-merger market share as $(1, \ldots, n)$. If $P(\Delta \Pi|J, A^C(J)) = \emptyset$, then $\Delta \Pi \leq \Delta \Pi(1, \beta_1(Q^\circ))$. From Proposition 1, it follows immediately that $P(\Delta \Pi|J, A^*(\Delta \Pi|J)) = \emptyset$. Hence, properties (25) and (26) hold for set $J_n$.

So suppose now instead that $P(\Delta \Pi|J, A^C(J)) \neq \emptyset$. Note that, since $\Delta \Pi(k, \pi_k^C(J))$ is increasing in $k$, the set $P(\Delta \Pi|J, A^C(J))$ is of the form $P(\Delta \Pi|J, A^C(J)) = \{1, \ldots, j(\Delta \Pi)\}$ for some $j(\Delta \Pi)$.

Consider first the treatment of mergers with target 1. We have $\pi_1^*(J) = \beta_1(Q^\circ)$. Moreover, the following two properties hold for all $\Delta \Pi$: for any $A^*(\Delta \Pi|J_n) \in \arg \max_{A \subseteq \Pi \times \Pi_j} \text{ECS}(\Delta \Pi; A, J_n)$,

$$1 \in P(\Delta \Pi|J_n, A^*(\Delta \Pi|J_n)) \Leftrightarrow 1 \in P(\Delta \Pi|J_n, A^C(J_n))$$

(27)

and

$$A^*_1(\Delta \Pi|J_n) = A^C_1(J_n) \text{ if } 1 \in P(\Delta \Pi|J_n, A^C(J_n)).$$

(28)

These follow from the following facts: (i) No CS-decreasing merger $M_1$ can be accepted in $A^*(\Delta \Pi|J_n)$; (ii) for any $A \subseteq \Pi_j \times \Pi_j$, $\text{ECS}(\Delta \Pi(1, \tau_1); A, J_n) < \Delta CS(1, \tau_1)$ for all $\tau_1 < \beta_1(Q^\circ)$, so all mergers $M_1 = (1, \tau_1)$ such that $\tau_1 < \beta_1(Q^\circ)$ and $\Delta \Pi(1, \tau_1) \leq \Delta \Pi$ must be in $A^*(\Delta \Pi|J_n)$, and (iii) accepting all mergers $M_1$ such that $\Delta \Pi(1, \tau_1) > \Delta \Pi$ maximizes $\Pr(\Delta \Pi(M_1) > \Delta \Pi$ and $M_1 \in A_1)$ and, since accepting the mergers described in (ii) is optimal, therefore maximizes $\text{ECS}(\Delta \Pi; A, J_n)$.

Now, consider a merger with target $k > 1$ and assume:
Proof of Proposition 3.

Induction Hypothesis 2: For all \( k' < k \), the following two properties hold for all \( \Delta \Pi \): for any \( A^*(\Delta \Pi|J_n) \in \arg \max_{A \subseteq \Pi_{j \notin J_n}[1, h]} ECS(\Delta \Pi; A, J_n) \),

\[
k' \in P(\Delta \Pi|J_n, A^*(\Delta \Pi|J_n)) \iff k' \in P(\Delta \Pi|J_n, A^C(J_n))
\]

and

\[
A^*_k(\Delta \Pi|J_n) = A^C_k(J_n) \text{ if } k' \in P(\Delta \Pi|J_n, A^C(J_n)).
\]

We will show that properties (29) and (30) hold as well for \( k \) so that Induction Hypothesis 2 holds for \( k + 1 \). Suppose, first, that \( k \notin P(\Delta \Pi|J_n, A^C(J_n)) \). Then every \( M_k \) with \( \Delta \Pi(M_k) \leq \Delta \Pi \) has \( ECS(\Delta \Pi(M_k); A^C_{j_n \setminus k}(J_n), J_n \setminus k) > \Delta CS(M_k) \). But by Induction Hypothesis 2 and Proposition 1 [which implies that in \( A^*(\Delta \Pi|J_n) \) we must have \( \Delta \Pi_k < \Delta \Pi_j \) for any \( j \) such that \( j \in P(\Delta \Pi|J_n, A^*(\Delta \Pi|J_n)) \)], which implies that \( ECS(\Delta \Pi(M_k); A^C_{j_n \setminus k}(J_n), J_n \setminus k) = ECS(\Delta \Pi(M_k); A^C_{j_n \setminus k}(J_n), J_n \setminus k). \)

Hence, merger \( M_k \) cannot be in \( A^*(\Delta \Pi|J_n) \); i.e., \( k \notin P(\Delta \Pi|J_n, A^*(\Delta \Pi|J_n)) \).

Suppose now instead that \( k \in P(\Delta \Pi|J_n, A^C(J_n)) \). Observe, first, that every \( M_k = (k, \bar{\pi}_k) \) with \( \bar{\pi}_k > \pi^C_k(J_n) \) has \( ECS(\Delta \Pi(M_k); A^C_{j_n \setminus k}(J_n), J_n \setminus k) > \Delta CS(M_k) \), and since by Induction Hypothesis 2 and Proposition 1, \( ECS(\Delta \Pi(M_k); A^*_n(\Delta \Pi|J_n), J_n \setminus k) = ECS(\Delta \Pi(M_k); A^C_{j_n \setminus k}(J_n), J_n \setminus k) \), the merger cannot be in \( A^*(\Delta \Pi|J_n) \); i.e., \( A_k^*(\Delta \Pi|J_n) \subseteq A^C_k(J_n) \). Next, consider mergers \( M_k = (k, \bar{\pi}_k) \) with \( \bar{\pi}_k < \pi^C_k(J_n) \). Condition (12) combined with Induction Hypotheses 1 and 2 imply that each of these mergers satisfies \( \Delta CS(M_k) > ECS(\Delta \Pi(M_k); A^C(J_n \setminus k), J_n \setminus k) = \max_{A \subseteq \Pi_{j \notin J_n}[1, h]} ECS(\Delta \Pi(M_k), A, J_n \setminus k), \)

and hence must be included in \( A^*(\Delta \Pi|J_n) \); i.e., \( A^C_k(J_n) \subseteq A^*_k(\Delta \Pi|J_n) \). We thus have \( A^C_k(J_n) = A^*_k(\Delta \Pi|J_n). \) Hence, properties (29) and (30) hold as well for \( k \). Applying induction (twice) then yields the result.

\[
\text{Proof of Proposition 3.} \text{ Let } A \text{ denote the optimal approval policy with cutoffs } (\pi_1, ..., \pi_K) \text{ when } Pr(\phi_k = 1) = \theta_k, \text{ and let } A' \text{ denote the optimal approval policy with cutoffs } (\pi'_1, ..., \pi'_K) \text{ when } Pr(\phi_k = 1) = \theta'_k. \text{ From the recursive definition of the cutoffs, it follows immediately that a change in } \theta_k \text{ does not affect the cutoffs for any smaller merger } M_j, j < k, \text{ nor the cutoff of merger } M_k \text{ itself. Hence, } \Delta CS' = \Delta CS \text{ for all } j \leq k.

Consider now the cutoff for merger \( M_{k+1}, k + 1 \leq K \). We can write the cutoff condition as

\[
\Delta CS_{k+1} = Pr(\phi_k = 1) \Delta \Pi (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) \leq \Delta \Pi(k + 1, \bar{\pi}_{k+1})
\]

\[
\times E_{\tilde{\Theta}(1,...,k)} [\Delta CS (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) | \Delta \Pi (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) \leq \Delta \Pi(k + 1, \bar{\pi}_{k+1}) \text{ and } \phi_k = 1]
\]

\[
+ [1 - Pr(\phi_k = 1)] \Delta \Pi (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) \leq \Delta \Pi(k + 1, \bar{\pi}_{k+1})
\]

\[
\times E_{\tilde{\Theta}(1,...,k)} [\Delta CS (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) | \Delta \Pi (M^*(\tilde{\Theta}(1,...,k), A_{\{1,...,k\}})) \leq \Delta \Pi(k + 1, \bar{\pi}_{k+1}) \text{ and } \phi_k = 0]
\],

where \( A_j \equiv \Pi_{j \in J} A_j \).
Note first that the optimal policy must be such that

\[
E_{\tilde{\phi}(1,\ldots,k)} [\Delta CS (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) | M_{k+1} = (k+1, \tilde{\alpha}_{k+1}),
\]
\[
\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (M_{k+1}), \text{ and } \phi_k = 1
\]
\[
> E_{\tilde{\phi}(1,\ldots,k)} [\Delta CS (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) | M_{k+1} = (k+1, \tilde{\alpha}_{k+1}),
\]
\[
\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (M_{k+1}), \text{ and } \phi_k = 0
\].

To see this, consider the case where \( \phi_k = 1 \) and \( \Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}) \).

Two cases can arise: (i) \( M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k)) \neq M_k \) and (ii) \( M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k)) = M_k \). In case (i) the outcome is the same as when \( M_k \) were not feasible (\( \phi_k = 0 \)). In case (ii), merger \( M_k \) will be implemented. If merger \( M_k \) were not feasible, we would instead obtain the expected consumer surplus of the next most profitable allowable merger. By the optimality of the approval policy, \( \Delta CS(M_k) \) must weakly exceed (and, generically, strictly) the expected consumer surplus of the next-most profitable allowable merger.

Next, note that we can rewrite the conditional probability as

\[
\Pr(\phi_k = 1 | \Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}))
\]
\[
= \Pr(\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}) | \phi_k = 1) \theta_k
\]
\[
\times \left\{ \Pr(\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}) | \phi_k = 0) (1 - \theta_k) \right\}^{-1}
\]
\[
= \left\{ 1 + \frac{\Pr(\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}) | \phi_k = 0)}{\Pr(\Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1}) | \phi_k = 1)} \right\}^{-1}
\].

Hence, an increase in \( \theta_k \) induces an increase in the conditional probability \( \Pr(\phi_k = 1 | \Delta \Pi (M^* (\tilde{\phi}(1,\ldots,k), A(1,\ldots,k))) \leq \Delta \Pi (k+1, \tilde{\alpha}_{k+1})) \). But this implies that an increase in \( \theta_k \) induces an increase in the RHS of the cutoff condition for merger \( M_{k+1} \). Hence, \( \Delta CS'_{k+1} > \Delta CS_{k+1} \).

Consider now the induction hypothesis that \( \Delta CS'_{k'} > \Delta CS_{k'} \) for all \( k < k' < j \). In particular, \( \Delta CS'_{j-1} > \Delta CS_{j-1} \). We claim that this implies that \( \Delta CS'_{j} > \Delta CS_{j} \). To see this, note that we can decompose the effect of the increase in \( \theta_k \) on the conditional expectation of the next-most profitable merger into two steps:

1. Increase the feasibility probability from \( \theta_k \) to \( \theta'_k \), holding fixed the approval policy \( A \).

2. Change the approval policy from \( A \) to \( A' \).

Consider first step (1). For the same reason as before, the increase in the feasibility probability must raise the conditional expectation

\[
E_{\tilde{\phi}(1,\ldots,j-1)} [\Delta CS (M^* (\tilde{\phi}(1,\ldots,j-1), A(1,\ldots,j-1))) | \Delta \Pi (M^* (\tilde{\phi}(1,\ldots,j-1), A(1,\ldots,j-1))) \leq \Delta \Pi (j, \tilde{\alpha}_j)]
\]

by the optimality of the approval policy \( A \).
Consider now step (2). The outcome under the two policies differs only in the event where \( M^* \left( \tilde{\mathcal{F}}_{1,\ldots,i-1}, \mathcal{A}_{1,\ldots,i-1} \right) \notin \mathcal{A}' \). Let \( M_i = M^* \left( \tilde{\mathcal{F}}_{1,\ldots,j-1}, \mathcal{A}_{1,\ldots,j-1} \right) \). Under policy \( \mathcal{A} \), the outcome in this event is \( \Delta CS(M_i) \). Under policy \( \mathcal{A}' \) instead, the expected outcome is

\[
E_{\tilde{\mathcal{F}}_{1,\ldots,i-1}} \left[ \Delta CS \left( M^* \left( \tilde{\mathcal{F}}_{1,\ldots,i-1}, \mathcal{A}'_{1,\ldots,i-1} \right) \right) \right] \mid \Delta \Pi \left( M^* \left( \tilde{\mathcal{F}}_{1,\ldots,i-1}, \mathcal{A}'_{1,\ldots,i-1} \right) \right) \leq \Delta \Pi (k, \tilde{\tau}_i) .
\]

But as \( M_i \notin \mathcal{A}' \), we must have

\[
E_{\tilde{\mathcal{F}}_{1,\ldots,i-1}} \left[ \Delta CS \left( M^* \left( \tilde{\mathcal{F}}_{1,\ldots,i-1}, \mathcal{A}'_{1,\ldots,i-1} \right) \right) \right] \mid \Delta \Pi \left( M^* \left( \tilde{\mathcal{F}}_{1,\ldots,i-1}, \mathcal{A}'_{1,\ldots,i-1} \right) \right) \leq \Delta \Pi (j, \tilde{\tau}_i) > \Delta CS(M_i) .
\]

As the expected consumer surplus increases at each step, we must have \( \Delta CS'_j > \Delta CS_j \). □

References


Lemma 5. Consider the function $H(s_1, ..., s_N) = \sum_n (s_n)^2$ and two vectors $s' = (s'_1, ..., s'_N)$ and $s'' = (s''_1, ..., s''_N)$ having $\sum_{n=1}^N s'_n = \sum_{n=1}^N s''_n$. If for some $r$, (i) $s'_r \geq s''_j$ for all $j \neq r$, (ii) $s''_r > s'_r$, and (iii) $s''_j \leq s'_j$ for all $j \neq r$, then $H(s'') > H(s')$.

Proof. Without loss of generality, take $r = 1$ and define $\Delta_n \equiv s'_n - s''_n$ for $n > 1$. Observe that $\Delta_n \geq 0$ for all $n > 1$ and $\Delta_n > 0$ for some $n > 1$. Define as well the vectors $s'' = (s'_1 + \sum_{t=2}^{n} \Delta_t, s''_2 - \Delta_2, ..., s''_n - \Delta_n, s''_{n+1}, ..., s''_N)$ for $n > 1$ and $s' \equiv s'$. Note that $s'' = s''$. Then

$$H(s'') - H(s') = \sum_{n=1}^{N-1} [H(s''^{n+1}) - H(s^n)].$$

Now letting $\tilde{s}_1' \equiv s'_1$ and $\tilde{s}_1'' \equiv s'_1 + \sum_{t=2}^{n} \Delta_t \geq s''_1$ for all $n > 1$, each term in this sum is nonnegative,

$$H(s''^{n+1}) - H(s^n) = (\tilde{s}_1'' + \Delta_{n+1})^2 + (s''_{n+1} - \Delta_{n+1})^2 - (\tilde{s}_1'')^2 - (s''_{n+1})^2 = 2\Delta_{n+1}(\tilde{s}_1'' - s''_{n+1}) + 2(\Delta_{n+1})^2 \geq 0,$$

and strictly positive if $\Delta_{n+1} > 0$. Since $\Delta_{n+1} > 0$ for some $n \geq 1$, the result follows.\qed
8.1 Efficient Bargaining Among a Subset of Firms

Suppose instead that the outcome of the bargaining process maximizes the joint profit of only a subset of firms, \( L \), that includes firm 0 and all of the targets, i.e., \((\{0\} \cup \mathcal{K}) \subseteq L \subseteq \mathcal{N}\). That is, the proposal rule is

\[
M^* (\mathcal{F}, \mathcal{A}) \equiv \arg \max_{M_k \in (\mathcal{F} \cap \mathcal{A})} \Delta \Pi_L (M_k),
\]

where \( \Delta \Pi_L (M_k) \) now denotes the induced change in the joint profit of the firms in set \( L \), \( \Delta \Pi_L (M_k) \equiv \sum_{i \in \mathcal{L}\setminus \{0\}} \pi_i (M_k) - \sum_{i \in \mathcal{L}} \pi_i^0 \).

Under the same conditions as in the case of efficient bargaining, Proposition 1 carries over to this bargaining process. The key point is the following: If any CS-nondecreasing merger or any reduction in a merged firm’s marginal cost induces an increase in aggregate profit, then it also induces an increase in the joint profit of the firms in set \( L \). This follows because both a CS-nondecreasing merger and a reduction in a firm’s post-merger marginal cost weakly reduce the profit of any nonmerging firm, including the firm(s) not in set \( L \). This observation has several implications. First, it means that part (iv) of Lemma 4 continues to hold if we replace aggregate profit by the joint profit of the firms in set \( L \). Second, it also means that a reduction in the post-merger marginal cost \( \bar{c}_k \) raises the joint profit of the firms in set \( L \) for any CS-nondecreasing merger. Third, Lemma ?? continues to hold if we replace the induced change in aggregate profit by the induced change in the joint profit of the firms in \( L \). This follows because the two mergers in the statement of the lemma, \( M_j \) and \( M_k \), induce (by assumption) the same change in consumer surplus, so the profit of any firm \( i \neq j, k \) is the same under both mergers.

As a result, we can again draw a figure like Figure 1, and all of the steps in the proof of Proposition 1 carry over to this case.

8.2 Differentiated Products

In our analysis we have assumed that firms produce a homogeneous good and compete in a Cournot fashion. Focusing on the case of efficient bargaining between firms, we now show that our main insights carry over to the case where firms compete in prices and produce symmetrically differentiated goods with consumers having CES or multinomial logit demand. These models share with the Cournot model an important property: they are “aggregative games.” Using this common structure, we show below that if merger \( M_k \) is CS-neutral, then it raises the joint profit of the merging firms as well as aggregate profit. Moreover, a reduction in post-merger marginal cost increases the merged firm’s profit and, provided pre-merger differences between firms are not too large, aggregate profit. Moreover, if any two mergers \( M_j \) and \( M_k \), \( k > j \), induce the same nonnegative change in consumer surplus, then the larger merger \( M_k \) induces a greater increase in aggregate profit than the smaller merger \( M_j \). In sum, in the two differentiated goods models, the merger curves have the same features in \((\Delta CS, \Delta \Pi)\)-space as in the Cournot model. Our main result, Proposition 1, therefore carries over as well.
Assumptions. Suppose an unmerged firm $i$’s profit can be written as
\[ \pi(\psi_i; c_i; \Psi), \]
where $\psi_i \geq 0$ is firm $i$’s strategic variable, $c_i$ the firm’s constant marginal cost, and $\Psi \equiv \sum_j \psi_j$ an aggregator summarizing the “aggregate outcome.” The firm’s cumulative best reply, $r(\Psi; c_i) \equiv \arg \max \psi_i \pi(\psi_i; c_i; \psi_i + \sum_{j \neq i} \psi_j)$, is assumed to be single-valued and decreasing in marginal cost $c_i$.

Similarly, a merged firm $k$’s profit is given by $2\pi(\psi_k, \tau_k; \Psi)$, and its cumulative best reply, $\tau(\Psi; \tau_k) \equiv \arg \max \psi_i 2\pi(\psi_k, \tau_k; 2\psi_k + \sum_{j \neq 0, k} \psi_j)$, is single-valued and decreasing in $\tau_k$. Consumer surplus, denoted $V(\Psi)$, is an increasing function of the aggregator and does not depend on the composition of the aggregator.

Suppose that there exists a unique stable equilibrium. Let $\psi_i(M_k)$ denote firm $i$’s equilibrium action under market structure $M_k$, and $\Psi(M_k) \equiv \sum_j \psi_j(M_k)$. Further, suppose that firm $i$’s equilibrium profit can be written as
\[ g(\psi_i(M_k); \Psi(M_k)) \equiv \max_{\psi_i} \pi(\psi_i, c_i; \Psi(M_k)) \text{ if firm } i \text{ is unmerged}; \]
\[ g(2\psi_i(M_k); \Psi(M_k)) \equiv \max_{\psi_i} 2\pi(\psi_i, c_i; \Psi(M_k)) \text{ if firm } i = k \text{ is merged}. \]

The equilibrium profit function $g$ has the following properties: (i) $g(0; \Psi) = 0$; (ii) for $0 \leq \psi_i \leq \Psi$, $g(\psi_i; \Psi)$ is strictly increasing and strictly convex in $\psi_i$. We assume that a reduction in post-merger marginal cost $\tau_k$ leads to (a) an increase in $\psi_k(M_k)$ and in the aggregate outcome $\Psi(M_k)$; (b) an increase in $\psi_k(M_k)/\Psi(M_k)$ and a decrease in $\psi_j(M_k)/\Psi(M_k)$, $j \neq 0, k$; and (c) an increase in the merged firm’s equilibrium profit $g(2\psi_k(M_k), \Psi(M_k))$ and a reduction in any other firm $i$’s equilibrium profit $g(\psi_i(M_k); \Psi(M_k))$.

Our assumptions hold for several textbook models of competition.

Example 3 (Cournot). In the homogeneous goods Cournot model with constant marginal costs, let $\psi_i$ denote the output of plant $i$. All unmerged firms can be thought of as single-plant firms, whereas a merged firm can be thought of as running two plants at the same marginal cost (producing the same output at both plants). We impose the same assumptions on demand as in the main text. The profit maximization problem of a single-plant firm $i$ with marginal cost $c_i$ can be written as
\[ \max_{\psi_i} P(\psi_i + \sum_{j \neq i} \psi_j) - c_i \psi_i. \]

From the first-order condition of profit maximization, $P(\Psi) - c_i + \psi_i P'(\Psi) = 0$, we can write the equilibrium profit under merger $M_k$ as
\[ g(\psi_i(M_k); \Psi(M_k)) = -[\psi_i(M_k)]^2 P'(\Psi(M_k)). \]

The profit maximization problem of a merged firm $k$ with marginal cost $\tau_k$ (and two plants) can be written as
\[ \max_{\psi_k} P(2\psi_k + \sum_{j \neq 0, k} \psi_j) - \tau_k. \]
From the first-order condition of profit maximization, $P(\Psi) - \tau_k + 2\psi_k P'(\Psi) = 0$, so that we can write the merged firm’s equilibrium profit under merger $M_k$ as

$$g(2\psi_k(M_k); \Psi(M_k)) = -\left[2\psi_k(M_k)\right]^2 P'(\Psi(M_k)).$$

It can easily be verified that $g$ has all of the required properties (it takes the value of zero if its first argument is zero and is increasing and convex in its first argument) and that a reduction in post-merger marginal cost $\tau_k$ has the posited effects. (The other assumptions were shown to hold in the main text.)

**Example 4** (CES). In the CES model, the utility function of the representative consumer is given by

$$U = \left( \sum_{i=0}^{N} X_i^\rho \right)^{1/\rho} Z^\alpha,$$

where $\rho \in (0, 1)$ and $\alpha > 0$ are parameters, $X_i$ is consumption of differentiated good $i$, and $Z$ is consumption of the numeraire. Utility maximization implies that the representative consumer spends a constant fraction $1/(1 + \alpha)$ of his income $Y$ on the $N + 1$ differentiated goods (and the remainder on the numeraire). Using the normalization $Y/(1 + \alpha) \equiv 1$, the resulting demand for differentiated good $i$ is

$$X_i = \frac{p_i^{-\lambda - 1}}{\sum_{j=0}^{N} p_j^{-\lambda}},$$

where $p_i$ is the price of good $i$, and $\lambda \equiv \rho/(1 - \rho)$. The consumer’s indirect utility can be written as

$$V = (1 + \alpha) \ln Y + \frac{1}{\lambda} \ln \left( \sum_{j=0}^{N} p_j^{-\lambda} \right). \quad (31)$$

Suppose that firms compete in prices, and that an unmerged firm produces a single good, while a merged firm produces two goods at the same marginal cost (thus optimally charging the same price for each). Consider first a single-product firm $i$. The profit maximization problem of a single-product firm $i$ with marginal cost $c_i$ can be written as

$$\max \left[ \psi_i^{-1/\lambda} - c_i \right] - \frac{\psi_i^{(\lambda+1)/\lambda}}{\psi_i + \sum_{j \neq i} \psi_j}.$$

From the first-order condition of profit maximization,

$$-\Psi + \left[ \psi_i^{-1/\lambda} - c_i \right] \psi_i^{(\lambda+1)/\lambda} \left\{ \frac{(\lambda + 1)\Psi}{\psi_i} - \lambda \right\} = 0,$$

it can be seen that there is a unique cumulative best reply $r(\Psi; c_i)$ and that it is decreasing in the firm’s marginal cost $c_i$. We can write the firm’s equilibrium profit under merger $M_k$ as

$$g (\psi_i(M_k); \Psi(M_k)) \equiv \left\{ \frac{(\lambda + 1)\Psi(M_k)}{\psi_i(M_k)} - \lambda \right\}^{-1}.$$

Consider now the merged firm $k$ and suppose the firm produces two products at marginal cost $\bar{\tau}_k$. The profit maximization problem can be written as

$$\max 2[\psi_k^{-1/\lambda} - \bar{\tau}_k] \frac{\psi_k^{(\lambda+1)/\lambda}}{2\psi_k + \sum_{j \neq 0,k} \psi_j}.$$
(It can easily be verified that the firm optimally chooses the same value of $\psi_k$ for each one of its two products.) From the first-order condition,

$$-\Psi + \left[ \psi_k^{-1/\lambda} - \tau_k \right] \psi_k^{(\lambda+1)/\lambda} \left\{ \frac{(\lambda + 1)\Psi}{\psi_k} - 2\lambda \right\} = 0,$$

it can be seen that there is a unique cumulative best reply $r(\Psi; \tau_k)$ and that it is decreasing in $\tau_k$. We can write the merged firm’s equilibrium profit under merger $M_k$ as

$$g(2\psi_k(M_k); \Psi(M_k)) \equiv \left\{ \frac{(\lambda + 1)\Psi(M_k)}{2\psi_k(M_k)} - \lambda \right\}^{-1}.$$ 

It can easily be verified that our assumptions hold in the CES model. In particular, there exists a unique equilibrium and this equilibrium is stable.\(^{21}\) Moreover, the equilibrium profit function $g$ has all of the required properties (it takes the value of zero if its first argument is zero and is increasing and convex in its first argument). Consider a reduction in post-merger marginal cost $\tau_k$. Since $r(\Psi; \tau_k)$ is decreasing in $\tau_k$ and since $r(\Psi; c_i)$ and $\tau(\Psi; \tau_k)$ are increasing in $\Psi$, and since equilibrium is stable, the reduction in $\tau_k$ induces a higher value of $\Psi = 2r(\Psi; \tau_k) + \sum_{i \neq 0, k} r(\Psi; c_i)$. Rewrite the first-order condition of an unmerged firm $i$:

$$-1 + \left[ 1 - c_i [r(\Psi; c_i)]^{1/\lambda} \right] \left\{ (\lambda + 1) - \lambda \frac{r(\Psi; c_i)}{\Psi} \right\} = 0.$$

As the induced increase in $\Psi$ induces an increase in $r(\Psi; c_i)$ (i.e., prices are strategic complements), the ratio $\frac{r(\Psi; c_i)}{\Psi}$ must fall as otherwise the l.h.s. of the first-order condition would decrease. But as

$$\frac{2r(\Psi; \tau_k)}{\Psi} + \sum_{i \neq 0, k} r(\Psi; c_i) \Psi = 1,$$

it follows that the same ratio for the merged firm, $\frac{\tau(\Psi; \tau_k)}{\Psi}$, must increase. From the expression for the equilibrium profits, we thus obtain that the profit of the merged firm, $g(2r(\Psi(M_k); \tau_k); \Psi(M_k))$, increases and that of any unmerged firm $i$, $g(r(\Psi(M_k), c_i); \Psi(M_k))$, decreases.

**Example 5 (Multinomial Logit).** In the multinomial logit model, expected demand for product $i$ is given by

$$X_i = \frac{\exp \left( \frac{a - p_i}{\mu} \right)}{\sum_{j=0}^{N} \exp \left( \frac{a - p_j}{\mu} \right)}.$$ 

\(^{21}\)From the first-order condition for profit maximization, we obtain that $dr(\Psi; c_i)/d\Psi$ can be written as a decreasing and convex function of $\beta_i \equiv \Psi/r(\Psi; c_i)$:

$$\frac{dr(\Psi; c_i)}{d\Psi} = \frac{\lambda}{(\lambda + 1)\beta_i(\beta_i - 1) + \lambda}.$$ 

This derivative attains its maximum of 1 if firm $i$ is the only active firm (i.e., $r(\Psi; c_i) = \Psi$). [Similarly, for a merged firm $M_k$, we have

$$\frac{d2r(\Psi; \tau_k)}{d\Psi} = \frac{\lambda}{(\lambda + 1)\beta_k(\beta_k - 1) + \lambda},$$

where $\beta_k \equiv \Psi/[2r(\Psi; \tau_k)]$.] It follows that $\sum_i dr(\Psi; c_i)/d\Psi < 1$ [resp. $\sum_{i \neq 0, k} dr(\Psi; c_i)/d\Psi + 2r(\Psi; \tau_k)/d\Psi < 1$ after merger $M_k$] in any equilibrium with more than one active firm. Hence, any equilibrium must be stable. Moreover, as $r(0; c_i) \geq 0$ [resp. $\tau(0; \tau_k) \geq 0$] and $r(\Psi; c_i) = 0$ [resp. $\tau(\Psi; \tau_k) = 0$] for $\Psi$ sufficiently large, this implies that there exists a unique $\Psi$ that is consistent with equilibrium in the sense that $\Psi - \sum_i r(\Psi; c_i) = 0$ [resp. $\Psi - \sum_{i \neq 0, k} r(\Psi; c_i) - 2r(\Psi; \tau_k) = 0$ after merger $M_k$].
where \( a > 0 \) and \( \mu > 0 \) are parameters, and \( p_j \) the price of product \( j \). Letting \( Y \) denote income, the indirect utility of the representative consumer can be written as

\[
V = Y + \mu \ln \left( \sum_{j=0}^{N} \exp \left( \frac{a - p_j}{\mu} \right) \right).
\]  

(32)

Suppose that firms compete in prices, and that an unmerged firm produces a single good, while a merged firm produces two goods at the same marginal cost (and optimally charging the same price for each). Consider first a single-product firm \( i \). The profit maximization problem of a single-product firm \( i \) with marginal cost \( c_i \) can be written as

\[
\max_{\psi_i} \left[ a - \mu \ln \psi_i - c_i \right] \frac{\psi_i}{\psi_i + \sum_{j \neq i} \psi_j}.
\]

From the first-order condition of profit maximization,

\[
\{ -\mu + a - \mu \ln \psi_i - c_i \} \Psi - \left[ a - \mu \ln \psi_i - c_i \right] \psi_i = 0,
\]

it can be seen that there is a unique cumulative best reply \( r(\Psi; c_i) \) and that it is decreasing in the firm’s marginal cost \( c_i \). Firm \( i \)’s equilibrium profit under merger \( M_k \) can be written as

\[
g(\psi_i(M_k); \Psi(M_k)) = \mu \left( \frac{\Psi(M_k)}{\psi_i(M_k)} - 1 \right)^{-1}.
\]

Consider now the merged firm \( k \) and suppose the firm produces two products at marginal cost \( \tau_k \). The profit maximization problem can be written as

\[
\max_{\psi_k} 2 \left[ a - \mu \ln \psi_k - \tau_k \right] \frac{\psi_k}{2\psi_k + \sum_{j \neq 0,k} \psi_j}.
\]

(It can easily be verified that the firm optimally chooses the same value of \( \psi_k \) for each one of its two products.) From the merged firm’s first-order condition of profit maximization,

\[
\{ -\mu + a - \mu \ln \psi_k - \tau_k \} \Psi - 2 \left[ a - \mu \ln \psi_k - \tau_k \right] \psi_k = 0,
\]

it can be seen that there is a unique cumulative best reply \( r(\Psi; \tau_k) \) and that it is decreasing in \( \tau_k \). Firm \( k \)’s equilibrium profit under merger \( M_k \) can be written as

\[
g(2\psi_k(M_k); \Psi(M_k)) = \mu \left( \frac{\Psi(M_k)}{2\psi_k} - 1 \right)^{-1}.
\]

It can easily be verified that our assumptions hold in the multinomial logit model. In particular, there exists a unique equilibrium and this equilibrium is stable.\(^{22}\) Moreover, the equilibrium profit function \( g \) has all of the required properties (it takes the value of zero if its first argument is zero and is increasing

\[^{22}\text{From the first-order condition for profit maximization, we obtain that } \frac{dr(\Psi; c_i)}{d\Psi} \text{ can be written as a decreasing and convex function of } \beta_i \equiv \Psi/r(\Psi; c_i): \]

\[
\frac{dr(\Psi; c_i)}{d\Psi} = \frac{1}{\beta_i(\beta_i - 1) + 1}.
\]

This derivative attains its maximum of 1 if firm \( i \) is the only active firm (i.e., \( r(\Psi; c_i) = \Psi \)). Similarly, for a merged firm
and convex in its first argument). Consider a reduction in post-merger marginal cost \( c_k \). Since \( \tau(\Psi; c_k) \) is decreasing in \( c_k \) and since \( r(\Psi; c_i) \) and \( \tau(\Psi; c_k) \) are increasing in \( \Psi \), and since equilibrium is stable, the reduction in \( c_k \) induces a higher value of \( \Psi = 2\tau(\Psi; c_k) + \sum_{i \neq k} r(\Psi; c_i) \). Rewrite the first-order condition of an unmerged firm \( i \):

\[
-1 + \left[ 1 - c_i [r(\Psi; c_i)]^{1/\lambda} \right] \left\{ (\lambda + 1) - \lambda \frac{r(\Psi; c_i)}{\Psi} \right\} = 0.
\]

As the induced increase in \( \Psi \) induces an increase in \( r(\Psi; c_i) \) (i.e., prices are strategic complements), the ratio \( r(\Psi; c_i)/\Psi \) must fall as otherwise the l.h.s. of the first-order condition would decrease. But as

\[
\frac{-\mu + a - \mu \ln r(\Psi; c_i) - c_i}{a - \mu \ln r(\Psi; c_i) - c_i} = \frac{r(\Psi; c_i)}{\Psi},
\]

it follows that the same ratio for the merged firm, \( \tau(\Psi; c_k)/\Psi \), must increase. From the expression for the equilibrium profits, we thus obtain that the profit of the merged firm, \( g(2\tau(\Psi(M_k); c_k); \Psi(M_k)) \), increases and that of any unmerged firm \( i \), \( g(r(\Psi(M_k); c_i); \Psi(M_k)) \), decreases.

Results. Let \( \psi^0_i \equiv \psi_i(M_0) \) and \( \psi^\circ \equiv \Psi(M^\circ) \), and note that, as consumer surplus \( V(\Psi) \) is strictly increasing in \( \Psi \), merger \( M_k \) is CS-neutral if \( \Psi(M_k) = \psi^\circ \); it is CS-increasing if \( \Psi(M_k) > \psi^\circ \), and CS-decreasing if \( \Psi(M_k) < \psi^\circ \).

Lemma 6. Merger \( M_k \) is CS-neutral if \( \psi^0_k = \psi^0 + \psi^k \), CS-increasing if \( \psi^0_k > \psi^0 + \psi^k \), and CS-decreasing if \( \psi^0_k < \psi^0 + \psi^k \).

Proof. Suppose merger \( M_k \) is CS-neutral. Then, \( \Psi(M_k) = \psi^\circ \). From the profit maximization problem of any firm \( i \) not involved in the merger, it follows that \( \psi_i(M_k) = r(\Psi(M_k); c_i) = \psi_i^\circ \). Hence, we must have \( 2\psi_k(M_k) = \psi^0 + \psi^k \). The claim then follows from the observation that consumer surplus is increasing in \( \Psi \) and that the equilibrium is stable. \( \square \)

Lemma 7. If merger \( M_k \) is CS-neutral, it raises the joint profit of the merging firms as well as aggregate profit.

Proof. It is immediate to see that the profit of any firm not involved in the merger remains unchanged as \( \Psi \) remains unchanged. It thus remains to show that

\[
g(2\psi_k(M_k); \Psi(M_k)) > g(\psi^0; \psi^\circ) + g(\psi^k; \psi^\circ).
\]

For merger \( M_k \), we have

\[
\frac{d2\tau(\Psi; c_k)}{d\Psi} = \frac{1}{2\tau(c_k)} \left[ \frac{d\tau(\Psi; c_k)}{d\Psi} - 1 \right],
\]

where \( \tau(c_k) = \Psi/[2\tau(\Psi; c_k)] \). It follows that \( \sum_{i} dr(\Psi; c_i)/d\Psi < 1 \) [resp. \( \sum_{i \neq 0,k} dr(\Psi; c_i)/d\Psi + 2\tau(\Psi; c_k)/d\Psi < 1 \) after merger \( M_k \)] in any equilibrium with more than one active firm. Hence, any equilibrium must be stable. Moreover, as \( r(0; c_i) \geq 0 \) [resp. \( r(0; c_k) \geq 0 \) and \( r(\Psi; c_i) = 0 \) [resp. \( r(\Psi; c_k) = 0 \) for \( \Psi \) sufficiently large, this implies that there exists a unique \( \Psi \) that is consistent with equilibrium in the sense that \( \Psi = \sum_{i} r(\Psi; c_i) = 0 \) [resp. \( \Psi - \sum_{i \neq 0,k} r(\Psi; c_i) - 2\tau(\Psi; c_k) = 0 \) after merger \( M_k \)].

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But as $M_k$ is CS-neutral, we have $\Psi(M_k) = \Psi^o$ and $2\psi_k(M_k) = \psi_k^o + \psi_k^o$. The above inequality can thus be rewritten as

$$g(\psi^o + \psi_k^o; \Psi^o) > g(\psi_k^o; \Psi^o) + g(\psi_k^o; \Psi^o).$$

But this follows from the assumed properties of the function $g$. \hfill $\square$

As a reduction in post-merger marginal cost increases the merged firm’s profit, any CS-nondecreasing merger is profitable. As in the Cournot model with efficient bargaining, we impose the following assumption:

**Assumption 3.** If merger $M_k$, $k \geq 1$, is CS-nondecreasing, then reducing its post-merger marginal cost $\tau_k$ increases the aggregate profit

$$\left[g(2\psi_k(M_k); \Psi(M_k)) + \sum_{i \in \mathcal{N} \setminus \{0, k\}} g(\psi_i(M_k); \Psi(M_k))\right].$$

In the CES and multinomial logit models (and, as we have seen before, in the Cournot model), a sufficient condition for this assumption to hold is that pre-merger cost differences are not too large so that for every merger $M_k$, $(\psi_k^o + \psi_k^o)/\Psi^o > \max_{i \neq 0, k} \psi_i^o/\Psi^o$, i.e., the sum of the pre-merger shares of the merger partners exceeds the pre-merger share of the largest nonmerging firm.

**Example 6** (CES). In the CES model, if pre-merger marginal cost differences are not too large so that $(\psi_k^o + \psi_k^o)/\Psi^o > \max_{i \neq 0, k} \psi_i^o/\Psi^o$, then the reduction in post-merger marginal cost $\tau_k$ following a CS-nondecreasing merger $M_k$ increases aggregate profit. To see this, note that from the argument given in our exposition of the CES model above, the reduction in $\tau_k$ induces a change from $\psi_i/\Psi$ to $(\psi_i/\Psi - \Delta_i)$, $i \neq 0, k$, $\Delta_i > 0$, and from $2\psi_k/\Psi$ to $(2\psi_k/\Psi + \sum_{i \neq 0, k} \Delta_i)$. It thus suffices to show that the joint profit of the merged firm $k$ and any other firm $i$,

$$h_i(\Delta) = \left\{ \frac{\sigma_k + \Delta}{(\lambda + 1) - \lambda(\sigma_k + \Delta)} \right\} + \left\{ \frac{\sigma_i - \Delta}{(\lambda + 1) - \lambda(\sigma_i - \Delta)} \right\},$$

where $\Delta \in [0, \Delta_i]$, $\sigma_i = \psi_i/\Psi$ and $2\psi_k/\Psi \leq \sigma_k \leq 2\psi_k/\Psi + \sum_{i \neq 0, k} \Delta_j$, is increasing in $\Delta$. But this holds as we have

$$h_i'(\Delta) = \frac{\lambda + 1}{[(\lambda + 1) - \lambda(\sigma_k + \Delta)]^2} - \frac{\lambda + 1}{[(\lambda + 1) - \lambda(\sigma_i - \Delta)]^2} > 0,$$

where the inequality follows since $\psi_k^o + \psi_k^o > \psi_i^o$ implies that $\sigma_k > \sigma_i$ for any CS-nondecreasing merger $M_k$.

**Example 7** (Multinomial Logit). In the multinomial logit model, if pre-merger marginal cost differences are not too large so that $(\psi_k^o + \psi_k^o)/\Psi^o > \max_{i \neq 0, k} \psi_i^o/\Psi^o$, then the reduction in post-merger marginal cost $\tau_k$ following a CS-nondecreasing merger $M_k$ increases aggregate profit. To see this, note that from the argument given in our exposition of the multinomial logit model above, the reduction in $\tau_k$ induces a change from $\psi_i/\Psi$ to $(\psi_i/\Psi - \Delta_i)$, $i \neq 0, k$, $\Delta_i > 0$, and from $2\psi_k/\Psi$ to $(2\psi_k/\Psi + \sum_{i \neq 0, k} \Delta_i)$. It thus suffices to show that the joint profit of the merged firm $k$ and any other firm $i$,

$$h_i(\Delta) = \mu \left\{ \frac{\sigma_k + \Delta}{1 - (\sigma_k + \Delta)} \right\} + \mu \left\{ \frac{\sigma_i - \Delta}{1 - (\sigma_i - \Delta)} \right\},$$

and any other firm $j$, where $h_i(\Delta)$ and $h_j(\Delta)$ are both increasing in $\Delta$. This holds as we have

$$h_i'(\Delta) = \frac{\lambda + 1}{[(\lambda + 1) - \lambda(\sigma_k + \Delta)]^2} - \frac{\lambda + 1}{[(\lambda + 1) - \lambda(\sigma_i - \Delta)]^2} > 0,$$

where the inequality follows since $\psi_k^o + \psi_k^o > \psi_i^o$ implies that $\sigma_k > \sigma_i$ for any CS-nondecreasing merger $M_k$.\hfill $\square$
where \( \Delta \in [0, \Delta_1] \), \( \sigma_i = \psi_i/\Psi \) and \( 2\psi_k/\Psi \leq \sigma_k \leq 2\psi_k/\Psi + \sum_{j \neq 0,k} \Delta_j \), is increasing in \( \Delta \). But this holds as we have

\[
K_1' (\Delta) = \frac{\mu}{[1 - (\sigma_k + \Delta)]^2} - \frac{\mu}{[1 - (\sigma_i - \Delta)]^2} > 0,
\]

where the inequality follows since \( \psi^i_0 + \psi^o > \psi^o_i \) implies that \( \sigma_k > \sigma_i \) for any CS-nondecreasing merger \( M_k \).

We are now in the position to extend Lemma 4 to this larger class of models:

**Lemma 8.** Suppose mergers \( M_j \) and \( M_k, k > j \), induce the same nonnegative change in consumer surplus so that \( \Psi(M_j) = \Psi(M_k) \geq \Psi^o \). Then, the larger merger \( M_k \) induces a greater increase in aggregate profit than the smaller merger \( M_j \).

**Proof.** As the aggregate outcome \( \Psi \) is the same under both mergers, the profit of each firm not participating in either merger is also the same under both mergers. We thus only need to show that

\[
g(2\psi_k(M_k); \Psi) + g(\psi_j(M_k); \Psi) > g(2\psi_j(M_j); \Psi) + g(\psi_k(M_j); \Psi),
\]

where \( \Psi = \Psi(M_j) = \Psi(M_k) \) is the common aggregate outcome after each of the two alternative mergers. As \( \Psi(M_j) = \Psi(M_k) \), we must have

\[
2\psi_k(M_k) + \psi_j(M_k) = 2\psi_j(M_j) + \psi_k(M_j).
\]

Now, as \( c_j > c_k \) and as \( \Psi(M_j) = \Psi(M_k) \), we obtain (from the assumption that a firm’s cumulative best reply is decreasing in its marginal cost) that

\[
\psi_j(M_k) < \psi_k(M_j),
\]

implying that

\[
2\psi_k(M_k) > 2\psi_j(M_j).
\]

Next, note that as a CS-nondecreasing merger increases the profit of the merging firms and reduces everyone else’s profit, we have

\[
g(2\psi_k(M_k), \Psi(M_k)) > g(\psi^o_k, \Psi^o) \geq g(\psi_k(M_j), \Psi(M_j)).
\]

As \( \Psi(M_k) = \Psi(M_j) \) and as \( g \) is strictly increasing in its first argument, this implies that

\[
2\psi_k(M_k) > \psi_k(M_j).
\]

Using the same type of argument, we also have

\[
2\psi_j(M_j) > \psi_j(M_k).
\]
We have thus shown that

\[ 2\psi_k(M_k) > \max \{2\psi_j(M_j), \psi_k(M_j)\} \geq \min \{2\psi_j(M_j), \psi_k(M_j)\} > \psi_j(M_k). \]

But since \(2\psi_k(M_k) + \psi_j(M_k) = 2\psi_j(M_j) + \psi_k(M_j)\) and since \(g\) is strictly convex in its first argument, this implies that

\[ g(2\psi_k(M_k); \overline{\Psi}) + g(\psi_j(M_k); \overline{\Psi}) > g(2\psi_j(M_j); \overline{\Psi}) + g(\psi_k(M_j); \overline{\Psi}). \]

Finally, note that if \(|\overline{\Psi} - \Psi^\circ|\) is sufficiently small, where \(\overline{\Psi} \equiv \Psi(M_j) = \Psi(M_k) \geq \Psi^\circ\), then the lemma also implies that the larger merger \(M_k\) induces a larger increase in the bilateral profit change than the smaller merger \(M_j\). (This follows from the fact that if both mergers are CS-neutral, then the induced bilateral profit change is equal to the induced aggregate profit change.)

### 8.3 General Sets of Mergers

So far, we have assumed that there is a single firm, firm 0, that is part of every potential merger. Moreover, we have assumed that every merger involves only two firms, firm 0 and one target. In this section, we relax both of these assumptions by allowing for general sets of mergers. As the offer game no longer seems an appropriate bargaining process once there is no single firm that is party to every potential merger, we focus on efficient bargaining. We continue to assume that at most one merger can be proposed to the antitrust authority. We provide sufficient conditions under which the main result of the paper carries over to this more general setting. In particular, we show that the key criterion according to which the antitrust authority should optimally discriminate between alternative mergers is the naively-computed post-merger Herfindahl index. This naively-computed post-merger index is frequently used by antitrust authorities in merger analysis as it is entirely based on readily available information on pre-merger market structure.

To proceed, let \(m_k \geq 2\) denote the number of merger partners in merger \(M_k\) and let \(c_{M_k}\) denote the realized post-merger marginal cost of merger \(M_k\). It is straightforward to see that the characterization of CS-neutral mergers in Lemma 1 extends to any \(m_k \geq 2\). In particular, any CS-neutral merger raises aggregate profit. In the main text, we have shown that aggregate profit following merger \(M_k\) is proportional to the post-merger Herfindahl index \(H(M_k)\), where the proportionality factor depends only on the post-merger aggregate output \(Q(M_k)\) [see (10)]. Observe that for a CS-neutral merger \(M_k\), Lemma 1 implies that the actual post-merger Herfindahl index equals the naively-computed index:

\[
H(M_k) = \frac{|s_k(M_k)|^2 + \sum_{i \in M_k} |s_i(M_k)|^2}{\sum_{i \in M_k} s_i^0} + \sum_{i \notin M_k} [s_i^0]^2 \equiv H^{naive}(M_k).
\]
Thus, for any two CS-neutral mergers $M_j$ and $M_k$, regardless of the number of merger partners, the merger that induces a greater naively-computed post-merger Herfindahl index also induces a greater increase in aggregate profit:

$$H^{naive}(M_k) > H^{naive}(M_j) \Leftrightarrow \Delta \Pi(M_k) > \Delta \Pi(M_j).$$

Hence, provided that merger curves slope upward in the positive orthant of $(\Delta \Pi, \Delta CS)$-space and do not intersect, Proposition 1 carries over to this more general setting, where a “larger” merger now refers to a merger that induces a greater increase in the naively-computed post-merger Herfindahl index.

Under what conditions do the curves for CS-nondecreasing mergers slope upward and not intersect? To identify such conditions, we first state the following result:

**Lemma 9.** The slope of the curve for merger $M_k$ in $(\Delta \Pi, \Delta CS)$-space is given by

$$\frac{d\Delta \Pi(M_k)}{d\Delta CS(M_k)} = -2 - \left[ \frac{P''(Q(M_k))Q(M_k)}{P'(Q(M_k))} \right] H(M_k) + \left[ \frac{2}{P'(Q(M_k))Q(M_k)} \right] r(Q(M_k); \bar{c}_{M_k}),$$

where $r(Q; c) \equiv \{ q | P(Q) - c + qP'(Q) = 0 \}$ is the “cumulative best reply” of a firm with marginal cost $c$ to aggregate output $Q$.

**Proof.** The change in $\Delta CS$ induced by a small increase in post-merger marginal cost is

$$\frac{d\Delta CS(M_k)}{d\bar{c}_{M_k}} = -P'(Q)Q \frac{d\bar{Q}}{d\bar{c}_{M_k}},$$

where $\bar{Q} \equiv Q(M_k)$ is aggregate output following merger $M_k$. Recall that aggregate profit can be written as $\eta(\bar{Q})H$, where $H \equiv H(M_k)$ is the post-merger Herfindahl index and $\eta(\bar{Q}) \equiv -P'(\bar{Q})\bar{Q}^2$. The effect of a small increase in post-merger marginal cost on the change in aggregate profit induced by merger $M_k$ is thus given by

$$\frac{d\Delta \Pi(M_k)}{d\bar{c}_{M_k}} = \eta'(\bar{Q}) \frac{d\bar{Q}}{d\bar{c}_{M_k}} H + \eta(\bar{Q}) \frac{dH}{d\bar{c}_{M_k}},$$

where

$$\eta'(\bar{Q}) = [-P''(\bar{Q})\bar{Q}^2 + 2P'(\bar{Q})\bar{Q}].$$

Putting this together, we obtain

$$\frac{d\Delta \Pi(M_k)}{d\Delta CS(M_k)} = -\left[ \frac{\eta'(\bar{Q})}{P'(\bar{Q})\bar{Q}} \right] H - \left[ \frac{\eta(\bar{Q})}{P'(\bar{Q})\bar{Q}} \right] \left( \frac{dH/d\bar{c}_{M_k}}{d\bar{Q}/d\bar{c}_{M_k}} \right)$$

$$= \left[ 2 + \frac{P''(\bar{Q})\bar{Q}}{P'(\bar{Q})} \right] H + \bar{Q} \left( \frac{dH/d\bar{c}_{M_k}}{d\bar{Q}/d\bar{c}_{M_k}} \right).$$

(34)
Now, we have
\[
\frac{dH}{d\bar{c}_{M_k}} = \frac{d}{d\bar{c}_{M_k}} \left[ \sum_i r(Q; c_i) \right] = \left( \frac{2}{Q^3} \right) \frac{dQ}{d\bar{c}_{M_k}} \left[ \sum_i r(Q; c_i)^2 \right] + \left( \frac{1}{Q^2} \right) \left[ 2r(Q; \bar{c}_{M_k}) \frac{\partial r(Q; \bar{c}_{M_k})}{\partial \bar{c}_{M_k}} + 2 \sum_i r(Q; c_i) \frac{dr(Q; c_i)}{dQ} \frac{dQ}{d\bar{c}_{M_k}} \right] \\
= - \left( \frac{2}{Q^2} \right) H \frac{dQ}{d\bar{c}_{M_k}} + \left( \frac{2}{P'(Q)Q^2} \right) r(Q; \bar{c}_{M_k}) - \left( \frac{2}{Q^2} \right) \frac{dQ}{d\bar{c}_{M_k}} \sum_i \left[ r(Q; c_i) + \frac{P''(Q)}{P'(Q)} r(Q; c_i)^2 \right] \\
= - \left( \frac{2}{Q} \right) H \frac{dQ}{d\bar{c}_{M_k}} + \left( \frac{2}{P'(Q)Q^2} \right) r(Q; \bar{c}_{M_k}) - \left( \frac{2}{Q^2} \right) \frac{dQ}{d\bar{c}_{M_k}} - 2 \frac{P''(Q)}{P'(Q)} H \frac{dQ}{d\bar{c}_{M_k}},
\]

where the third equality follows using the facts that \( \partial r(Q; \bar{c}_{M_k})/\partial \bar{c}_{M_k} = 1/P''(Q) \) and \( dr(Q; c_i)/dQ = - (1 + r(Q; c_i)P''(Q)/P'(Q)) \). Thus, we have:
\[
\frac{Q(dH/d\bar{c}_{M_k})}{(dQ/d\bar{c}_{M_k})} = -2H - 2 \left( \frac{P''(Q)Q}{P'(Q)} \right) H + \left( \frac{2}{P'(Q)Q} \right) r(Q; \bar{c}_{M_k}).
\]

Substituting (36) into (34), we obtain equation (33).

We now use expression (33) to identify conditions under which the merger curves are upward-sloping and non-intersecting.\(^{23}\) As earlier, merger curves are upward-sloping in the positive orthant whenever the pre-merger joint market share of any merging firms exceed the pre-merger share of the largest nonmerging firm. However, expression (33) allows us to derive a weaker condition than this:\(^{24}\)

**Lemma 10.** The merger curve of merger \( M_k \) slopes upward in the positive orthant of \((\Delta \Pi, \Delta CS)\)-space if the merged firm’s naively-computed post-merger market share \( s_{M_k}^{\text{naive}} = \sum_{i \in M_k} s_i^0 \) and the naively-computed post-merger Herfindahl index \( H_{M_k}^{\text{naive}} \) satisfy
\[
s_{M_k}^{\text{naive}} \geq \frac{H_{M_k}^{\text{naive}}}{2} \geq 1 - (N - m_k + 2)s_{M_k}^{\text{naive}},
\]

where \( N + 1 \) is the pre-merger number of firms (and thus \( N - m_k + 2 \) is the number of firms following merger \( M_k \)).

\(^{23}\)Condition (33) also offers an alternative method to establish Lemma 4. To see this, observe that, in our baseline model, if two mergers induce the same change in consumer surplus, \( \Delta CS \), and the same change in aggregate profit, \( \Delta \Pi \), then the two mergers also induce the same aggregate output \( Q \) and the same post-merger Herfindahl index \( H \). Moreover, in our baseline model, the firm resulting from a larger merger has a larger output \( r(Q; \bar{c}_M) \) (as it faces a larger \( \sum_{i \neq M} c_i \), and so must have a lower \( \bar{c}_M \) if it induces an equal CS-level). Hence, (33) implies in that model that if there were a point of intersection, the curve of the larger merger would have a larger value of \( d\Delta \Pi/d\Delta CS \), hence a flatter curve, which yields a contradiction since the larger merger’s curve must cross from below at the first crossing since the larger merger’s curve lies further to the right where \( \Delta CS = 0 \).

\(^{24}\)That condition (37) below holds when \( s_{M_k}^{\text{naive}} \geq \max_{i \in M_k} s_i^0 \) follows from the facts that in this case \( H_{M_k}^{\text{naive}} \leq s_{M_k}^{\text{naive}} \) and \( (N - m_k + 2)s_{M_k}^{\text{naive}} \geq 1 \). (Note that, in general, the Herfindahl index is bounded above by the share of the largest firm.)
Proof. Let $Q \equiv Q(M_k)$ denote post-merger aggregate output. Inserting

$$\frac{dQ}{d\tilde{c}_{M_k}} = \frac{1}{(N - m_k + 3)P'(Q) + \overline{Q}P''(\overline{Q})}$$

into equation (33), we obtain

$$\frac{d\Delta \Pi(M_k)}{d\Delta CS(M_k)} = -2 - \frac{\overline{Q}P''(\overline{Q})}{P'(\overline{Q})} \overline{\Pi} + \frac{2\bar{\sigma}_{M_k}}{P'(\overline{Q})} [(N - m_k + 3)P'(Q) + \overline{Q}P''(\overline{Q})]$$

$$= 2 [(N - m_k + 3)\bar{\sigma}_{M_k} - 1] + \frac{\overline{Q}P''(\overline{Q})}{P'(\overline{Q})} [2\bar{\sigma}_{M_k} - \overline{\Pi}],$$

where $\bar{\sigma}_{M_k}$ is the actual market share of the merged firm and $\overline{\Pi} \equiv H(M_k)$ the actual post-merger Herfindahl index.

Now, we claim that $s_{M_k}^{\text{naive}} \geq H^{\text{naive}}(M_k)/2$ implies that $2\bar{\sigma}_{M_k} \geq \overline{\Pi}$. To see this, note that the naively-computed inequality $s_{M_k}^{\text{naive}} \geq H^{\text{naive}}(M_k)/2$ corresponds to the case of a CS-neutral merger. As merger $M_k$ is CS-nondecreasing by assumption, it involves a (weakly) lower level of $\bar{\sigma}_{M_k}$ (and a weakly greater level of aggregate output) than a CS-neutral merger. It therefore suffices to show that a small reduction in $\bar{\sigma}_{M_k}$ leads to a larger value of $[2\bar{\sigma}_{M_k} - \overline{\Pi}]$, i.e., $d [2\bar{\sigma}_{M_k} - \overline{\Pi}] > 0$. But we have

$$d [2\bar{\sigma}_{M_k} - \overline{\Pi}] = 2d\bar{\sigma}_{M_k} - 2 \left( \bar{\sigma}_{M_k} d\bar{\sigma}_{M_k} + \sum_{i \notin M_k} \bar{\sigma}_i d\bar{\sigma}_i \right)$$

$$= 2(1 - \bar{\sigma}_{M_k}) \left( 1 - \sum_{i \notin M_k} d\bar{\sigma}_i \right) - 2 \sum_{i \notin M_k} \bar{\sigma}_i d\bar{\sigma}_i$$

$$> 0,$$

where the inequality follows from the observation that the induced increase in aggregate output reduces the market share of each nonmerging firm $i$, i.e., $d\bar{\sigma}_i < 0$.

Since Assumption 1 implies that $\overline{Q}P''(\overline{Q})/P'(\overline{Q}) > -1$, we obtain

$$\frac{d\Delta \Pi(M_k)}{d\Delta CS(M_k)} \geq 2 [(N - m_k + 3)\bar{\sigma}_{M_k} - 1] - [2\bar{\sigma}_{M_k} - \overline{\Pi}]$$

$$= 2 [(N - m_k + 2)\bar{\sigma}_{M_k} - 1] + \overline{\Pi}.$$

The r.h.s. of the last equation is positive if and only if

$$\frac{\overline{\Pi}}{2} \geq 1 - (N - m_k + 2)\bar{\sigma}_{M_k}.$$

We claim that this inequality is implied by the naively-computed analog,

$$\frac{H^{\text{naive}}(M_k)}{2} \geq 1 - (N - m_k + 2)s_{M_k}^{\text{naive}}.$$
To see this, consider the effect of decreasing the post-merger marginal cost \( \bar{\tau}_M \) on \( \bar{H} - 2(1 - (N - m_k + 2)\bar{\tau}_M) \):

\[
d \left[ \bar{H} - 2(1 - (N - m_k + 2)\bar{\tau}_M) \right] = 2 \left( \bar{\tau}_M d \bar{s}_M + \sum_{i \notin M_k} \bar{s}_i d \bar{s}_i \right) + 2(N - m_k + 2)d \bar{s}_M
\]

\[
= 2(N - m_k + 2 + \bar{\tau}_M) \left( 1 - \sum_{i \notin M_k} d \bar{s}_i \right) + 2 \sum_{i \notin M_k} \bar{s}_i d \bar{s}_i
\]

\[
= 2 \left\{ (N - m_k + 2 + \bar{\tau}_M) - \sum_{i \notin M_k} (N - m_k + 2 + \bar{\tau}_M - \bar{s}_i) d \bar{s}_i \right\}
\]

\[
> 0,
\]

where the inequality follows from the observation that \( d \bar{s}_i < 0 \) for all \( i \notin M_k \) and \( N - m_k + 2 \geq 1 \). \( \Box \)

Now consider when the merger curves are non-intersecting. We will use expression (33) to provide conditions under which two merger curves cannot cross; that is, their ranking must be the same as their ranking where \( \Delta CS = 0 \). We will show this by contradiction, showing that the curve further to the right at \( \Delta CS = 0 \) must have a smaller slope wherever the two curves cross. Since aggregate profit and consumer surplus are the same wherever the curves cross, so must be the industry Herfindahl index and aggregate quantity. By (33), this means the slopes at that point are ordered by the values of \( r(Q(M_k) ; \bar{\tau}_M_k) / (dQ(M_k) / d \bar{s}_M_k) \). The following lemma provides a condition under which those quantities are ordered in the correct way to give us an analog of Lemma 4:

**Lemma 11.** Consider two mergers \( M_j \) and \( M_k \), with \( m_j \geq m_k \). If the firms in \( M_k \) jointly produce more pre-merger than the firms in \( M_j \) (i.e., \( \sum_{i \in M_k} s_i > \sum_{i \in M_j} s_i \)) and if the naively-computed post-merger Herfindahl index is larger when \( M_k \) is implemented than when \( M_j \) is implemented [i.e., \( H^{\text{naive}}(M_k) > H^{\text{naive}}(M_j) \)], then the curve relating to merger \( M_k \) lies to the right of that relating to merger \( M_j \) in the positive orthant of \( (\Delta \Pi, \Delta CS) \)-space.

**Proof.** Let \( q^o \equiv \sum_{i \in M_l} q_i^o \), \( l = j, k \). The pre-merger first-order conditions imply that

\[
[m_l P(Q^o) - \sum_{i \in M_l} c_i] + P'(Q^o)q^o_{M_l} = 0 \text{ for } l = j, k,
\]

so

\[
(m_k - m_j)P(Q^o) - \sum_{i \notin M_k} c_i + \sum_{i \in M_k} c_i = P'(Q^o)(q^o_{M_k} - q^o_{M_j}) > 0. \tag{38}
\]

Next, summing up the post-merger first-order conditions, we have

\[
(N - m_l + 2)P(Q) - \sum_{i \in M_l} c_i + \sum_{i \in M_l} c_i - \bar{\tau}_{M_l} + P'(Q)Q = 0 \text{ for } l = j, k, \tag{39}
\]

where \( N + 1 \) is the number of firms prior to any merger. So,

\[
- \left[ (m_k - m_j)P(Q) - \sum_{i \notin M_k} c_i + \sum_{i \in M_k} c_i \right] - [\bar{\tau}_{M_k} - \bar{\tau}_{M_j}] = 0.
\]
Since \(-(m_k - m_j)P(Q) \leq -(m_k - m_j)P(Q)\) as the mergers are CS-nondecresing by assumption, we have

\[
- \left[ (m_k - m_j)P(Q) - \sum_{i \in M_k} c_i + \sum_{i \in M_j} c_i \right] - [\bar{\tau}_{M_k} - \bar{\tau}_{M_j}] \geq 0,
\]

so (38) implies that \(\bar{\tau}_{M_k} - \bar{\tau}_{M_j} < 0\), which in turn implies that \(r(Q; \bar{\tau}_{M_k}) > r(Q; \bar{\tau}_{M_j})\).

Applying the implicit function theorem to (39), yields

\[
\frac{dQ}{dc_{M_k}} = \frac{1}{(N - m_{l} + 3)P'(Q) + \bar{Q}P''(Q)}.
\]

As \(m_k \leq m_j\), \(P'(Q) < 0\), \(P''(Q) + \bar{Q}P''(Q) < 0\), and (from above) \(r(Q; \bar{\tau}_{M_k}) > r(Q; \bar{\tau}_{M_j})\), we obtain

\[
- \frac{r(Q; \bar{\tau}_{M_k})}{dQ/d\bar{c}_{M_k}} > \frac{r(Q; \bar{\tau}_{M_j})}{dQ/d\bar{c}_{M_j}}.
\]

Equation (33) implies that \(d\Delta \Pi / d\Delta CS\) is larger for merger \(M_k\) than for \(M_j\) at any point where the curves cross, from which the assertion follows.

Finally, if all mergers have the same minimum of the support of post-merger marginal costs, denoted \(l\), the maximum CS-increase that the smaller merger can achieve is larger than that of the larger merger.\(^{25}\) Hence, under the assumptions of Lemmas 10 and 11, the merger curves have all of the properties required to obtain our main result, the analog of Proposition 1.

For example, one special case in which this result can be applied arises where there are three potential mergers, one involving firms 1 and 2, a second involving firms 1 and 3, and a third involving firms 2 and 3. As before, the three mergers are mutually exclusive but, in contrast to the baseline model, there is no longer a single firm that is party to every potential merger. In this case, the two conditions of Lemma 11 are satisfied if the mergers have the same ranking by both the product and the sum of the merging firms’ pre-merger market shares.

### 8.4 Synergies in Fixed Costs

So far, we have assumed that firms have constant returns, implying that all merger-specific efficiencies involve marginal cost savings. We now consider the case where firms have to incur fixed costs, part of which may be saved by merging, and identify conditions under which our main result carries over to this setting. The discussion that follows applies to our baseline (offer-game) model as well to other scoring-rule bargaining models discussed in the article, with \(\Delta \Pi\) appropriately reinterpreted.

\(^{25}\)To see this, consider a larger merger \(M_k\) and a smaller merger \(M_j\), \(j < k\). The maximum \(\Delta CS\) induced by the larger merger \(M_k\) is \(\Delta CS(k, l)\). The assumption in Lemma 11 that the firms in \(M_k\) produce more pre-merger implies that \(\sum_{i \in M_k} c_i < \sum_{i \in M_j} c_i\). Since aggregate quantity depends only on the sum of firms’ costs, this in turn implies that if the two mergers induce the same change in consumer surplus, then \(\bar{\tau}_{M_k} < \bar{\tau}_{M_j}\). Thus, \(\Delta CS(k, l) = \Delta CS(j, \bar{\tau}_{M_j})\) implies that \(\bar{\tau}_{M_j} > l\). Since \(\Delta CS(j, \bar{\tau}_{M_j})\) is decreasing in \(\bar{\tau}_{M_j}\), the maximum CS-increase for the smaller merger, \(\Delta CS(j, l)\), must be larger than that of the larger merger: i.e., \(\Delta CS(j, l) > \Delta CS(k, l)\).
Figure 8: The figure depicts merger bands when mergers create both marginal and fixed cost savings in panel (a) and a possible approval set in panel (b).

Let $f_i$ denote the fixed cost of firm $i$ and assume that it is small enough that firm $i$ remains active following any merger by other firms. A feasible merger $M_k$ is now described by $M_k = (k, \pi_k, \overline{J}_k)$, where $\overline{J}_k \in [\overline{J}_k^l, \overline{J}_k^u] \subset \mathbb{R}_+$ is the realization of its post-merger fixed cost. The merger induces a fixed cost saving if $f_0 + f_k - \overline{J}_k = \alpha_k > 0$. Graphically, a fixed cost saving shifts the merger curve in a parallel fashion (by the amount of the saving) to the right in $(\Delta \Pi, \Delta CS)$-space. Thus, the possibility of fixed cost savings implies that the merger curves in $(\Delta \Pi, \Delta CS)$-space are now “broad bands,” with each point in the band of merger $M_k$ corresponding to a different realization of $(\pi_k, \overline{J}_k)$, and with the horizontal width of the band given by $|\overline{J}_k^u - \overline{J}_k^l|$ at any $\Delta CS(M_k)$. Figure 8 depicts the merger band for merger $M_k$.

When a feasible merger is proposed, the antitrust authority can observe all aspects of that merger, including the induced fixed cost saving. The antitrust authority’s approval set is now described by $A_k \equiv \{ M_k : (\pi_k, \overline{J}_k) \in A_k \} \cup M^o$, where $A_k \subseteq [l_k, h_k] \times [\overline{J}_k^l, \overline{J}_k^u]$. Without loss of generality, we restrict attention to approval sets that are regular in the sense that every $A_k$ is the closure of its interior, i.e., $A_k = \text{cl}(\text{int}(A_k))$. Let $\pi_k(\overline{J}_k) \equiv \max\{ \pi_k : (\pi_k, \overline{J}_k) \in A_k \}$ denote the largest allowable post-merger marginal cost level for a merger between firms 0 and $k$, conditional on the realized post-merger fixed cost $\overline{J}_k$. Let $\Delta CS_k(\overline{J}_k) \equiv \Delta CS(k, \pi_k(\overline{J}_k), \overline{J}_k)$ and $\Delta \Pi_k(\overline{J}_k) \equiv \Delta \Pi(k, \pi_k(\overline{J}_k), \overline{J}_k)$ denote the changes in consumer surplus and bilateral profits, respectively, induced by the “marginal merger” between firms 0 and $k$ given $\overline{J}_k$, and let $\Delta CS_k \equiv \min_{\overline{J}_k \in [\overline{J}_k^l, \overline{J}_k^u]} \Delta CS_k(\overline{J}_k)$ and $\Delta \Pi_k \equiv \min_{\overline{J}_k \in [\overline{J}_k^l, \overline{J}_k^u]} \Delta \Pi_k(\overline{J}_k)$ denote the lowest levels of $\Delta CS$ and $\Delta \Pi$, respectively, in any acceptable merger $M_k$. Figure 8(b) depicts an
approval set for merger $M_k$ and shows $\Delta CS_k$ and $\Delta \Pi_k$.

An immediate observation is the following. Suppose that fixed cost savings are nonnegative and perfectly correlated across mergers, so that $\alpha_k = \alpha \geq 0$ for every feasible merger $M_k \in \mathcal{F}$. Then the optimal approval set is constant in $\alpha$ in the sense that $(\tau_k, f_0 + f_k - \alpha) \in \mathcal{A}_k$ if and only if $(\tau_k, f_0 + f_k - \alpha') \in \mathcal{A}_k$, from which it follows that $\Delta CS_k(\bar{f}_k) = \Delta CS_k$ for all $\bar{f}_k$ and $k$. Moreover, as before, the optimal policy for any $\alpha$ is characterized by Proposition 1. To see this, note that the expected CS-maximizing antitrust authority cares about fixed cost savings only insofar as they affect firms’ merger proposals. But if fixed cost savings are perfectly correlated and nonnegative, the profit ranking of mergers (and teh profitability of CS-nondecreasing mergers) is unaffected by the fixed cost realization and all CS-nondecreasing mergers remain profitable.

Suppose now that the realized fixed cost saving of merger $M_k$ can be decomposed as follows:

$$\alpha_k = \alpha + \eta_k,$$

where $\alpha \in [\alpha^l, \alpha^h]$ is the (random or deterministic) component that is common across all feasible mergers (as above) and $\eta_k \in [\eta^l_k, \eta^h_k]$ is the component idiosyncratic to merger $M_k$. We assume that both the idiosyncratic shocks and post-merger marginal cost realizations are independent across mergers conditional on $\alpha$, have full support, and no mass points. We assume as well that when merger $M_k$ is proposed, the antitrust authority can observe $\alpha$ and $\eta_k$ separately (and condition the approval set on both components separately).\textsuperscript{26} Using the same arguments as above, it is straightforward to show that the optimal approval set is constant in $\alpha$. Therefore, for notational simplicity, we will from now on assume that there is no common component (i.e., $\alpha \equiv 0$), so that $\bar{f}_k = f_0 + f_k - \eta_k$.

In the remainder of this section, we assume that $|\bar{f}_k - \bar{f}_k|$ is sufficiently small so that the bands of the different mergers are non-overlapping in the positive orthant, as depicted in Figure 9. Thus, if any two mergers $M_j$ and $M_k$, $j < k$, induce the same nonnegative change in consumer surplus, then the larger merger is more profitable, regardless of the realized fixed cost savings. As fixed cost savings are nonnegative by assumption, the conclusion of Lemma 1 – that a CS-neutral merger is profitable – continues to hold.

Our main result, Proposition 1, carries over to this setting:

**Proposition 4.** In the model with fixed cost savings, any optimal approval policy $A$ approves the smallest merger if and only if it is CS-nondecreasing, approves only mergers $k \in K^+ \equiv \{1, \ldots, \bar{K}\}$ with positive probability ($\bar{K}$ may equal $K$), and satisfies $0 = \Delta CS_1 < \Delta CS_2 < \ldots < \Delta CS_{\bar{K}}$ for all $k \leq \bar{K}$.

**Proof.** Steps 1-2 proceed along the same lines as those in the proof of Proposition 1.

Step 3. As in the absence of fixed cost savings, any optimal policy has the property that, for all $k \in K^+$ and any $\bar{f}_k$, $\Delta CS_k(\bar{f}_k)$ is equal to the expected change in consumer surplus from the next-most
profitable merger \( M^* (\mathcal{F} \setminus (k, \pi_k(\mathcal{F}_k), \mathcal{I}_k), \mathcal{A}) \), conditional on the marginal merger \( M_k = (k, \pi_k(\mathcal{F}_k), \mathcal{I}_k) \) maximizing the change in the merging firms' bilateral profit in \( \mathcal{F} \cap \mathcal{A} \). That is,

\[
\Delta CS_k(\mathcal{F}_k) = E^k_\mathcal{F}[\Delta CS(M^* (\mathcal{F} \setminus M_k, \mathcal{A}))] M_k = (k, \pi_k(\mathcal{F}_k), \mathcal{I}_k) \text{ and } \Delta \Pi(M^* (\mathcal{F} \setminus M_k, \mathcal{A})) \leq \Delta \Pi(M_k)].
\]

To see that this equation must hold for all \( k \in \mathcal{K}^+ \), suppose first that \( \Delta CS_k(\mathcal{F}_k) > E^k_\mathcal{F}[\Delta CS(M^* (\mathcal{F} \setminus M_k, \mathcal{A}))] M_k = (k, \pi_k(\mathcal{F}_k), \mathcal{I}_k) \) for some firm \( k' \in \mathcal{K}^+ \) and fixed cost realization \( \mathcal{I}_{k'} \), and consider the alternative approval set \( \mathcal{A} \cup \mathcal{A}^c_{k'} \), where

\[
\mathcal{A}^c_{k'} \equiv \left\{ M_k : M_k = (k', \pi_{k'}(\mathcal{F}_{k'}), \mathcal{I}_{k'}) \text{ with } \pi_{k'} \in \left( \pi_{k'}(\mathcal{F}_{k'}), \pi_{k'}(\mathcal{F}_{k'}) + \varepsilon \right) \text{ and } \mathcal{I}_{k'} \in \left( \mathcal{I}_{k'} - \varepsilon, \mathcal{I}_{k'} + \varepsilon \right) \right\}.
\]

Using the same type of argument as in the proof of Proposition 1, it is straightforward to show that, for \( \varepsilon > 0 \) small enough, the change in expected consumer surplus from changing the approval set from \( \mathcal{A} \) to \( \mathcal{A} \cup \mathcal{A}^c_{k'} \) is strictly positive. A similar logic can be used to show that we cannot have \( \Delta CS_k(\mathcal{F}_{k'}) < E^k_\mathcal{F}[\Delta CS(M^* (\mathcal{F} \setminus M_k, \mathcal{A}))] \).

Step 4. Let \( \mathcal{M}^c_{j} \equiv \{ M_j : \Delta CS(M_j) = \Delta CS_j \text{ and } M_j \in \mathcal{A}_j \} \) denote the set of marginal mergers \( M_j \) that induce a change in consumer surplus of \( \Delta CS_j \), and let \( M^c_{j} \in \mathcal{M}^c_{j} \) denote the most profitable among these mergers, i.e., \( \Delta \Pi(M^c_{j}) \geq \Delta \Pi(M'_{j}) \) for all \( M'_{j} \in \mathcal{M}^c_{j} \). This merger is depicted in Figure 9.
10 for \( j = 2 \). An optimal acceptance set must have the property that, for all \( j < k \) such that \( j, k \in \mathcal{K}^+ \), we have \( \Delta \Pi(M_j^{CS}) \leq \Delta \Pi_k \). The argument is similar to (but slightly more involved than) Step 4 in the proof of Proposition 1: For \( j \in \mathcal{K}^+ \), let \( k' = \operatorname{arg\,min}_{k \in K^+, k > j} \Delta \Pi_k \) and suppose that, contrary to our claim, \( \Delta \Pi_{k'} < \Delta \Pi(M_{k'}^{CS}) \). In Figure 10 we suppose that \( k' = 3 \). Let \( M_k^{II} = (k', \pi_k' \langle \tilde{f}_{k'}^{II}, \tilde{f}_{k'}^{II} \rangle) \) denote the marginal merger \( M_k \) that induces the bilateral profit change \( \Delta \Pi_{k'} \), i.e., \( \Delta \Pi(M_k^{II}) = \Delta \Pi_{k'} \). By Step 3, \( M_k^{II} \) is uniquely defined, and \( \Delta \Pi_{k'} = E_k^{IA} \left( \pi_k' \langle \tilde{f}_{k'}^{II}, \tilde{f}_{k'}^{II} \rangle \right) \). Hence, by Step 3,

\[
\Delta \Pi = E_k^{IA} \left( \pi_k' \langle \tilde{f}_{k'}^{II}, \tilde{f}_{k'}^{II} \rangle \right) \geq \tau_2.
\]

Consider a change in the approval set from \( A \) to \( A \cup \mathcal{F}_j^\varepsilon \), where

\[
\mathcal{F}_j^\varepsilon = \{ M_j : \Delta \Pi(M_j) \in [\Delta \Pi_{k'} - \varepsilon, \Delta \Pi_{k'}] \},
\]

and \( \varepsilon > 0 \). Note that, as shown in Figure 10, \( \mathcal{F}_j^\varepsilon \subseteq A \). The change in expected consumer surplus from this change in the approval set equals \( \Pr(M^*(\mathcal{F}, A \cup \mathcal{F}_j^\varepsilon) \in (A \cup \mathcal{F}_j^\varepsilon) \setminus A) \) which is strictly positive as \( \mathcal{F}_j^\varepsilon \subseteq A \) times

\[
E_k^{IA} \left[ \Delta \Pi(M^*(\mathcal{F}, A \cup \mathcal{F}_j^\varepsilon)) - E_k^{A} (\tau_j, \tilde{f}_j) \right] M^*(\mathcal{F}, A \cup \mathcal{F}_j^\varepsilon) \in (A \cup \mathcal{F}_j^\varepsilon) \setminus A, \quad (41)
\]

where \( (\tau_j, \tilde{f}_j) \) is the pair of realized cost levels in the most profitable merger \( M^*(\mathcal{F}, A \cup \mathcal{F}_j^\varepsilon) \), which is a merger of firms 0 and \( j \) when the conditioning statement is satisfied. Now there exists a \( \delta > 0 \) such that for all \( \varepsilon > 0 \) the quantity in (41) is at least as large as

\[
E_k^{IA} \left[ \Delta \Pi(M_k^{II}) + \delta - E_k^{A} (\tau_j, \tilde{f}_j) \right] M^*(\mathcal{F}, A \cup \mathcal{F}_j^\varepsilon) \in (A \cup \mathcal{F}_j^\varepsilon) \setminus A. \quad (42)
\]

As \( \varepsilon \to 0 \), the quantity in (42) converges to \( [\Delta \Pi(M_k^{II}) - \tau_2] + \delta > 0 \), so for small enough \( \varepsilon > 0 \) (40) implies that this change in the acceptance set is strictly beneficial.

Step 5. For all \( j, k \in \mathcal{K}^+ \), \( j < k \), we must have \( \Delta \Pi_{j} < \Delta \Pi_{k} \). Suppose otherwise so that for some \( j, h \in \mathcal{K}^+ \), \( j > h \), we have \( \Delta \Pi_{j} \geq \Delta \Pi_{h} \). Let \( k \equiv \operatorname{arg\,min}[h \in \mathcal{K}^+ : h > j \text{ and } \Delta \Pi_{j} \geq \Delta \Pi_{h}] \). Figure 11 shows such a case where \( j = 2 \) and \( k = 3 \). Let merger \( M_k^{CS} \)’s marginal and fixed costs be \( \pi_k^{CS} = \pi_k(\tilde{f}_k^{CS}) \) and \( \tilde{f}_k^{CS} \), respectively. Given Step 4, \( E_k^{A} (\pi_k^{CS}, \tilde{f}_k^{CS}) \) can be written as a weighted average of two conditional expectations:

\[
E_k^{IA} \left[ \Delta \Pi(M^*(\mathcal{F}, M_k, A)) \right] M_k = (k, \pi_k^{CS}, \tilde{f}_k^{CS}), \quad M_k = M^*(\mathcal{F}, A), \text{ and } \Delta \Pi(M^*(\mathcal{F}, M_k, A)) < \Delta \Pi_j \quad (43)
\]
Figure 10: The figure illustrates the change considered in Step 4 of the proof of Proposition 4

and

\[ E_{\bar{\delta}} \left[ \Delta CS(M^*(\mathfrak{f}^k,M_k,A)) | M_k = (k, c_k^{CS}, T_k^{CS}) \right. \]
\[ \left. \text{and } \Delta \Pi(M^*(\mathfrak{f}^k,M_k,A)) \in [\Delta \Pi_j, \Delta \Pi(M_k)] \right] . \] (44)

Now the term in (43) equals \( E_{\bar{\delta}}^{k}(\bar{\pi}_k(T_k^{\Pi}), T_k^{\Pi}) \), which by Step 3 equals \( \Delta CS(M_j^{\Pi}) \), which in turn is at least \( \Delta CS_j \) by definition. On the other hand, the term in (44) strictly exceeds \( \Delta CS_j \). Together, this implies that \( E_{\bar{\delta}}^{k}(\bar{\pi}_k(T_k^{CS}), T_k^{CS}) > \Delta CS_j \). Since, by Step 3, we must have \( \Delta CS_k = E_{\bar{\delta}}^{k}(\bar{\pi}_k(T_k^{CS}), T_k^{CS}) \), this contradicts \( \Delta CS_j \geq \Delta CS_k \).

**Step 6.** The argument proceeds along the same lines as that in the proof of Proposition 1.

Thus, provided that idiosyncratic fixed cost synergies are small enough that merger bands do not overlap, it remains optimal to adopt a more stringent consumer surplus test for larger mergers. The restriction on the size of fixed cost synergies contrasts with the model of Armstrong and Vickers (2010). Their model, applied to the merger problem, assumes that the distribution of possible \( (\Delta \Pi, \Delta CS) \) pairs are the same for each merger and has a rectangular support. An interesting open question is how projects that are ex ante asymmetric in terms of their distribution of \( (\Delta \Pi, \Delta CS) \) pairs should be differentially treated when their supports overlap or even coincide.
Figure 11: The figure shows the change considered in Step 5 of the proof of Proposition 4